Display
D

Vol. 27, No. 4, March - April, 1954

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints, bound, ½¢ per page plus 10¢ each, (thus 25 ten page reprints would cost $1.25 plus $2.50 or $3.75) provided your order is placed before your article goes to press.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

Advertisers should contact The Lees Agencies, P.O. Box 906, Hollywood 28, California.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

MATHEMATICS MAGAZINE

Vol. 27, No. 4, March-April, 1954

CONTENTS

Extract from a Well-wishing Letter

"I am sincerely glad to see that this volume will be finished in the black. If there are any ways I can be of help in Mexico City please let me know. I will be very glad to be of any help."

Dear Friend:

Such letters as yours are in themselves very helpful. They give a human tone to work that might otherwise become very prosaic. It would help circulation if you would refer friends to articles in our Magazine which you feel would especially interest them. It would improve contents if more of our readers would submit for publication comments on published articles which especially interest them.

Cordially,
Glenn James

to various problems for which the stress field may be considered to consist of two uniform fields separated by a line of discontinuity. In chapter 6, incipient and steady plastic flows for which the shear lines may be approximated by a family of centered radial lines is discussed. Again, chapter 7 contains a discussion of a very interesting treatment by Galin (An analogy of the plane elastic-plastic problem, Prikladnaia Matematica i Mekhanika, 12, 1948) of the determination of the elastic-plastic boundary for a rectangular plate with a centered hole. The chapter closes with a treatment of the principle of virtual work, and the safety factor for incipient plastic flow. Here, theorems due to Drucker, Greenberg and Prager play a fundamental role in determining bounds on the safety factor. The problem of plane stress is noted in chapter 5 but is not treated. As the authors state, the difficulty is that the governing non-linear equation may change from hyperbolic to parabolic to elliptic type and hence from real to imaginary characteristics. It seems worthwhile to this reviewer to point out that for unrestricted plastic flow, the set of principal directions always exist and hence may furnish a powerful tool in analyzing this problem.

The final chapter is concerned with extremum principles; Cartesian tensor notation is used. By use of the divergence theorem and Schwarz's inequality, it is shown that certain integrals involving the stresses of strains attain their extremum for the actual stresses or strains. Thus, it appears possible that the direct methods of the calculus of variations may enable one to obtain further solutions to problems.

N. Coburn

Principles, methods of reasoning, and application are emphasized in H. C. Fryer's "Elements of Statistics," published in February by John Wiley & Sons, 263 pp., price $4.75.

Supplying a concise and general background in probability and statistics, Dr. Fryer gives the fundamentals applicable to all fields, without stressing any specialized area. He therefore draws extensively on examples from biology, economics, engineering, medicine, education, psychology, agriculture, and other fields that utilize statistical research.

After a brief history of the subject, the author covers the summarization of sets of data involving one type of measurement, elementary probability, and the binomial and normal frequency distribution. He also explains sampling from binomial populations, introductory sampling theory for a normal population involving only one variable, and linear regression and correlation.

John Wiley & Sons Richard Cook

SIMPLE OPERATIONAL EQUATIONS WITH CONSTANT COEFFICIENTS

D. C. Lewis

Foreword

A very elegant set of solutions of a system of n linear homogeneous differential equations with constant coefficients in n unknowns (each equation of the set being of order $\geqq 1$) is given by Frazer, Duncan, and Collar in their book, Elementary Matrices, Cambridge (1938), pp. 167-168, in terms of the columns of a certain matrix. But they do not prove that the general solution can be obtained by this method. The original purpose of this paper was to show that this is indeed the case. The problem reduces to the investigation of the rank of a certain complicated matrix (cf. §3).

The paper settles the question in the affirmative and at the same time shows that the same method applies equally well to other types of "operational equations," such as difference equations.

For an understanding of the paper a familiarity with calculus and matrix theory is required. Some understanding of functions of a complex variable would also be desirable.

§1. *Introduction.*

The object of this note is to give a unified treatment of systems of homogeneous linear differential and difference equations with constant coefficients. The work is also arranged so as to apply to other operational equations of a similar type.

All the results must be considered as essentially known,[1] although I know of no previous satisfactory treatment of systems of difference equations with constant coefficients. In the case of a system of differential equations written in the form $y' = Ay$, where A is a constant matrix and y is a vector, the solution in the form $y = exp(Ax)y_0$ is obvious; but, without a previous study of matrix functions, it is not clear that the matrix $exp(Ax)$ consists of elements that are linear in expressions of the form $exp(\lambda x)$ with coefficients that are in general polynomials in x. The matter is especially irksome if the elementary divisors of A are not simple. In the present paper the difficulties of this type (for difference systems as well as differential systems of a formally much more complicated nature) are isolated and given a succinct treatment in §3. The fundamental lemma in §3 is an elementary theorem in algebra, which I have never seen previously published, at least explicitly in this form.

In fact, the only place known to me, where such a question is treated, is in some work by Ernst Snapper,[2] who apparently obtained

from another point of view algebraic results far more abstract and general than those to be presented in this paper. In the first of Snapper's papers here cited, explicit application to linear systems of ordinary differential equations with constant coefficients is made (including systems containing more unknowns than independent equations, which are not treated in the present paper). In the second paper a generalization is made to partial differential equations (which are also not treated in the present paper) and their "exponential solutions." No explicit application is made to difference equations and other operational equations of the type here considered; but, since the algebraic theory of these systems is almost identical with that for systems of differential equations, it is practically certain that Snapper's results can also be applied to such systems of operational equations.

§2. *The system of operational equations and its solutions.*

Let C denote a linear class[3] of numerically valued functions $x(t)$ over which a linear[4] operator L is defined. The nature of the variable t and the set S of its values for which $x(t)$ is defined is arbitrary, except that it must be the same for each function in the class C. We suppose that L transforms every function of C into a function of C.[5] Suppose furthermore that for every complex value of λ in a certain region R of the complex plane, there is a function $\varphi_\lambda(t)\epsilon C$, which is analytic with respect to λ and satisfies the following five conditions:[6]

(I) $$\frac{d^p}{d\lambda^p}\varphi_\lambda(t)\epsilon C, \qquad p = 1, 2, \ldots$$

(II) $\varphi_\lambda(t) \neq 0$ for t in S and λ in R.

(III) The functions,

$$g_{p\lambda} = \frac{1}{\varphi_\lambda(t)}\frac{d^p}{d\lambda^p}\varphi_\lambda(t), \quad p = 0, 1, 2, \ldots .$$

are linearly independent of each other for each fixed value of λ i.e. There are no constants c_0, c_1, c_2, ... not all zero such that a finite sum of the type

$$\sum_p C_p g_{p\lambda}(t)$$

is identically zero.

(IV) $$L\varphi_\lambda(t) = \lambda\varphi_\lambda(t).$$

$$\text{(V)} \qquad L\,\frac{d^p\varphi_\lambda(t)}{d\lambda^p} = \frac{d^pL\varphi_\lambda(t)}{d\lambda^p}$$

Let $P(\lambda)$ be any polynomial

$$A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_m,$$

the coefficients being constants. The operator $P(L)$ is defined according to the usual convention in such a way that $P(L)\ x\ (t)$ is to denote

$$A_0L^n\ x\ (t) + A_1L^{n-1}\ x\ (t) + \cdots + A_n\ x\ (t).$$

Let $f(\lambda)$ represent a square matrix of order m whose elements are polynomials in λ. Let the element in the i^{th} row and j^{th} column be denoted by $f_{ij}(\lambda)$. Now consider the m operational equations

$$\text{(1)} \qquad \sum_{j=1}^{m} f_{ij}(L)x_j(t) = 0, \qquad i = 1, 2, \cdots, m,$$

in the m unknown functions

$$x_1(t), \cdots, x_m(t).$$

We shall also write these equations in matrix notation as follows:

$$\text{(1)} \qquad f(L)x(t) = 0.$$

We assume that $\Delta(\lambda) = \det. f(\lambda)$ is a polynomial of degree greater than zero. Suppose that the equation $\Delta(\lambda) = 0$ has an s-fold root λ_1 lying in the region R. Let $F(\lambda) = \operatorname{adj}. f(\lambda)$, i.e. the transposed of the matrix of the cofactors of $f(\lambda)$. Then *each column of each of the following s matrices yields a solution*[7] *of* (1):

$$\text{(2)} \qquad \frac{d^p}{d\lambda^p}\left[\varphi_\lambda(t)F(\lambda)\right]\Big|_{\lambda=\lambda_1} \qquad p = 0, 1, \cdots, s-1.$$

Here, of course, $\varphi_\lambda(t)$ represents a scalar and $F(\lambda)$ a matrix of order m. The proof of the italicized statement is surprisingly easy: Namely

$$f(L)\,\frac{d^p}{d\lambda^p}\left[\varphi_\lambda(t)F(\lambda)\right] = \frac{d^p}{d\lambda^p}\left[f(L)\varphi_\lambda(t)F(\lambda)\right] = \frac{d^p}{d\lambda^p}\left[\varphi_\lambda(t)f(\lambda)F(\lambda)\right].$$

Here we have made especial use of (V) and (IV). The last of the above expressions is

$$\frac{d^p}{d\lambda^p}\,[\varphi_\lambda(t)I\Delta(\lambda)]$$

which must vanish when $\lambda = \lambda_1$ and $p = 0, 1, 2, \cdots, s-1$, in as much as

$$\frac{d^p}{d\lambda^p}\,\Delta(\lambda)\,\Big|_{\lambda=\lambda_1}$$

vanishes, λ_1 being an s-fold **root** of the equation $\Delta(\lambda) = 0$.

§3. *The fundamental algebraic lemma.*

We shall eventually prove that there are just s linearly independent solutions to be had from the columns of the matrices (2). But we must first prove the following

LEMMA. *Let $f(z)$ represent a square matrix of order m whose elements are polynomials in z. Let $F(z)$ =* adj. *$f(z)$ and $\Delta(z)$ =* det. *$f(z)$, and lastly suppose that $\Delta(z)$ has a zero of order s (exactly) at the origin. Then the rank of the following square matrix of order ms is equal to s:*

$$\Phi = \left\|\begin{matrix} \binom{0}{0}F(0) & \binom{1}{0}F^{(1)}(0) & \binom{2}{0}F^{(2)}(0)\,\ldots\ldots & \binom{s-1}{0}F^{(s-1)}(0) \\ 0 & \binom{1}{1}F(0) & \binom{2}{1}F^{(1)}(0)\,\ldots\ldots & \binom{s-1}{1}F^{(s-2)}(0) \\ 0 & 0 & \binom{2}{2}F(0)\,\ldots\ldots & \binom{s-1}{2}F^{(s-3)}(0) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0\,\ldots\ldots\ldots\ldots & \binom{s-1}{s-1}F(0) \end{matrix}\right\|$$

Here, each entry in the above tabular display of the matrix Φ stands for a whole block of terms with m rows and m columns; the large 0's stand for such blocks of zeros. $F^{(p)}(z)$ is the p^{th} derivative of $F(z)$ and $\binom{p}{k}$ is the coefficient of b^k in the binomial expansion of $(1+b)^p$.

The slight change of notation from that used in the rest of the paper is introduced for convenience so as to have the root at the origin. The present section can be read independently of the rest of the paper.

Proof: An elementary transformation[8] of the matrix $f(z)$ may be proved to induce one or more elementary transformations of $F(z)$ and also of the large matrix Φ. Such transformations do not change the rank of Φ. Hence there is no loss of generality in assuming that $f(z)$ is a diagonal matrix, the element in the i^{th} row having the form

$$z^{\nu_i}\, \pi_i(z),$$

where

$$\pi_i(0) \neq 0, \quad \sum_{i=1}^{m} \nu_i = s \text{ and } \nu_i \leqq \nu_{i+1} \quad (i = 1, \cdots, m-1).$$

For, if $f(z)$ were not originally of this form, it could be put into this form by a series of elementary transformations. It follows that $F(z)$ is also a diagonal matrix, the element in the i^{th} row being of the form

$$z^{s-\nu_i} p_i(z),$$

where $p_i(0) \neq 0$. We therefore have $F(z) = W(z)P(z)$, where $W(z)$ is the diagonal matrix the element in whose i^{th} row is $z^{s-\nu_i}$ and $P(z)$ is another diagonal matrix, whose diagonal elements, $p_i(z)$, do not vanish for $z = 0$. By Leibnitz's rule for differentiating products, which holds also for matrices, we find

$$F^{(q)}(z) = \sum_{k=0}^{q} \binom{q}{k} W^{(q-k)}(z) P^{(k)}(z). \tag{3}$$

I say now that the matrix Φ has the same rank as the following matrix:

$$\Psi = \left\| \begin{matrix} \binom{0}{0}W(0) & \binom{1}{0}W^{(1)}(0) & \binom{2}{0}W^{(2)}(0) \ldots\ldots\ldots & \binom{s-1}{0}W^{(s-1)}(0) \\ 0 & \binom{1}{1}W(0) & \binom{2}{1}W^{(1)}(0) \ldots\ldots\ldots & \binom{s-1}{1}W^{(s-2)}(0) \\ 0 & 0 & \binom{2}{2}W(0) \ldots\ldots\ldots\ldots & \binom{s-1}{2}W^{(s-3)}(0) \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 \quad \ldots\ldots\ldots\ldots & \binom{s-1}{s-1}W(0) \end{matrix} \right\|$$

In fact the first m columns of Φ can be replaced by the first m columns of Ψ without changing the rank, in as much as the k^{th} column of Φ is $p_k(0)$ times the k^{th} column of Ψ $(k = 1, \ldots, m)$ and $p_k(0) \neq 0$. Let us now assume inductively that it is possible to replace the first mp columns of Φ by the corresponding mp columns of Ψ without change of rank. $(p = 1, 2, \ldots, s - 1)$. Let us call the resulting matrix Ω. We shall prove that it is then possible to replace the next m columns of Ω by the corresponding columns of Ψ without change of rank (i.e. columns $mp + 1, \ldots, mp + m$). The m-square block of elements in Ω which forms the intersection of columns $mp + 1, \ldots, mp + m$ and rows $mr + 1, \ldots, mr + m$ is represented by

$$\binom{p}{r} F^{(p-r)}(0).$$

In this connection we note that $\binom{p}{r} = 0$ if $r > p$.[9] From (3) we have

$$\binom{p}{r} F^{(p-r)}(0) = \sum_{k=0}^{p-r} \binom{p}{r}\binom{p-r}{k} W^{(p-r-k)}(0) P^{(k)}(0).$$

Since

$$\binom{p}{r}\binom{p-r}{k} = \binom{p}{k}\binom{p-k}{r} \text{ and } \binom{p-k}{r} = 0 \text{ for } k > p - r,$$

we obtain

$$\binom{p}{r} F^{(p-r)}(0) = \binom{p}{r} W^{(p-r)}(0) P(0) + \sum_{k=1}^{p} \binom{p-k}{r} W^{(p-k-r)}(0) \binom{p}{k} P^{(k)}(0).$$

The sum

$$\sum_{k=1}^{p}$$

on the right hand side of this matrix equation represents merely m linear combinations of the columns of the matrices

$$\binom{p-1}{r} W^{(p-1-r)}(0), \ldots, \binom{r}{r} W^{(0)}, \ldots, \binom{0}{r} W^{(-r)}(0),$$

the coefficients being independent of r. These matrices are blocks out of the first mp columns of Ω and from rows $mr + 1, \ldots, mr + m$. We thus see that by subtracting suitable linear combinations of the first mp columns of Ω from columns $mp + 1, \ldots, mp + m$, the blocks

$$\binom{p}{r} F^{(p-r)}(0)$$

in columns $mp + 1, \ldots, mp + m$ may be replaced by the blocks

$$\binom{p}{r} W^{(p-r)}(0)P(0).$$

This will not change the rank. Finally divide the $(mp + j)^{th}$ column $(j = 1, 2, \ldots, m)$ by its non-zero factor $p_j(0)$, and our proof by induction that the rank of Ψ is the same as the rank of Φ is complete.

It remains to prove that the rank of Ψ is equal to s. Let $w_{ij}(z)$ represent the element in the i^{th} row and j^{th} column of $W(z)$, and let $\delta(a,\beta) = 0$ if $a \neq \beta$ and $\delta(a, a) = 1$. Then $w_{ij}(z) = \delta(i, j)z^{s-\nu_i}$. Hence

$$w_{ij}^{(p)}(z) = \delta(i, j)(s - \nu_i)(s - \nu_i - 1)\ldots(s - \nu_i - p + 1)z^{s-\nu_i-p}$$

(4) $$w_{ij}^{(p)}(0) = \delta(i, j)\,\delta(s - \nu_i, p)(s - \nu_i)!.$$

Let us investigate those ms-matrices X that satisfy $\Psi X = 0$. Let the elements of X in the a^{th} column and the $(mp + k)^{th}$ row be denoted by $x_{p,k,a}$. $(p = 0, 1, \ldots, s - 1;\ k = 1, 2, \ldots, m;\ a = 1, 2, \ldots, ms)$. Then the matrix equation $\Psi X = 0$ stands for the following $(ms)^2$ scalar equations:

$$\sum_{p=0}^{s-1} \sum_{j=1}^{m} \binom{p}{r} w_{ij}^{(p-r)}(0)x_{p,j,\alpha} = 0,$$

$a = 1, 2, \ldots, ms;\ i = 1, 2, \ldots, m;\ r = 0, 1, \ldots, s - 1.$ Inserting the value of

$$w_{ij}^{(p-r)}(0)$$

given by (4) into the last equation, and dividing by $(s - \nu_i)!$, we find that

$$\sum_{p=0}^{s-1} \sum_{j=1}^{m} \binom{p}{r}\,\delta(i, j)\,\delta(s - \nu_i, p - r)x_{p,j,a} = 0.$$

(5) $$x_{s-\nu_i+r,i,a} = 0.$$

Suppose that $\mu_1, \ldots, \mu_\omega$ are the distinct values of the ν's; and suppose that k_l of the ν's are equal to μ_l, so that

$$\sum_{l=1}^{\omega} k_l\mu_l = \sum_{i=1}^{m} \nu_i = s.$$

If $\nu_i = \mu_l$, (5) becomes

$$x_{s-\mu_l+r,i,a} = 0, \quad l = 1, \ldots, \omega.$$

Here r may take on the μ_l values $0, 1, \ldots, \mu_l - 1$ and i takes on the k_l values for which $\nu_i = \mu_l$. For each a and for each value of l we thus get a certain set S_{la} of $k_l\mu_l$ of the quantities $x_{p,k,a}$ which must vanish. Now, if $\mu_l \neq \mu_h$, none of the values of i for which $\nu_i = \mu_l$ will be equal to a value of i for which $\nu_i = \mu_h$. Thus there will be no over lapping of the sets $S_{1a}, S_{2a}, \ldots, S_{\omega a}$. Hence there will be, for each value of a, exactly

$$\sum_{l=1}^{\omega} k_l\mu_l \ (= s) \text{ of the } x_{p,k,a}$$

which must vanish. Moreover the rows, in which these vanishing x's occur, are the same for each value of a. In other words, s rows of the matrix X must reduce to rows of zeros. The other elements of X are unrestricted. Hence X may be chosen to have the rank $ms - s$, but cannot be chosen to have a rank greater than $ms - s$. It follows that the rank of Ψ and hence of Φ is s, as we were to prove.

§4. *The linear relations which connect the solutions of the operational equations.*

Carrying out the differentiation indicated in (2) and using (III), we observe that each column of the matrix,

$$\sum_{k=0}^{p} \binom{p}{k} F^{(p-k)}(\lambda_1) g_{k\lambda_1}(t)\varphi_{\lambda_1}(t),$$

is a solution of (1). Here

$$F^{(q)}(\lambda_1) \text{ is the matrix } \frac{d^q}{d\lambda^q}F(\lambda)\Big|_{\lambda=\lambda_1}.$$

Since p assumes the s values $0, 1, 2, \ldots, s-1$ and since each matrix for a fixed p has m columns we have a total of ms solutions. But they are not independent of each other. Let $F_{ij}(\lambda)$ denote the element in the i^{th} row and j^{th} column of $F(\lambda)$. A linear relation connecting our solutions can be written in the form

$$\sum_{j=1}^{m}\sum_{p=0}^{s-1}\sum_{k=0}^{p} \binom{p}{k} F_{ij}^{(p-k)}(\lambda_1) g_{k\lambda_1}(t)\varphi_{\lambda_1}(t)A_{pj} \equiv 0, \quad i = 1, \ldots, m.$$

Dividing by $\varphi_{\lambda_1}(t)$ and changing the order of summation, we find that it is necessary and sufficient that the coefficients A_{pj} of our linear combination of solutions satisfy the relations

$$\sum_{k=0}^{s-1} \sum_{p=k}^{s-1} \sum_{j=1}^{m} \binom{p}{k} F_{ij}^{(p-k)}(\lambda_1) A_{pj} g_{k\lambda_1}(t) \equiv 0.$$

But since the $g_{k\lambda_1}(t)$ are linearly independent it is also both necessary and sufficient that

$$\sum_{p=k}^{s-1} \sum_{j=1}^{m} \binom{p}{k} F_{ij}^{(p-k)}(\lambda_1) A_{pj} = 0. \quad i = 1, \dots, m; \ k = 0, 1, \dots, s-1.$$

These relations represent a linear combination of the columns of a matrix of the type of Φ (discussed in 3) equated to zero. Since the rank of this matrix is s, there are just $ms - s$ independent linear relations (independent of t) connecting the columns of the s matrices (2); $p = 0, 1, \dots, s-1$. In other words, our method gives us just s linearly independent solutions corresponding to the s-fold root λ_1.

§5. *Concluding remarks. Examples.*

Suppose next that the equation $\Delta(\lambda) = 0$ has the distinct roots $\lambda_1, \dots, \lambda_N$ in R, the root λ_i occurring with multiplicity s_i. Corresponding to each root λ_i we have, according to §4, s_i linearly independent solutions. In order to conclude that we then have a total of

$$\sum_{i=1}^{N} s_i$$

linearly independent solutions, we must make a further hypothesis about the functions $\varphi_\lambda(t)$. The following condition is easily shown to be sufficient:

(VI) The functions

$$\frac{d^p}{d\lambda^p} \varphi_\lambda(t) \Big|_{\lambda=\lambda_i}, \qquad p = 0, 1, 2, \dots; \ i = 1, 2, \dots, N,$$

are linearly independent, $\lambda_1, \dots, \lambda_N$ denoting any N distinct numbers in R.

In fact, under this hypothesis, we argue as follows: Suppose there were a linear combination, with coefficients not all zero, of the

$$\sum_{i=1}^{N} s_i$$

solutions mentioned above, vanishing identically. We could write this as a sum of certain solutions σ_i, σ_j ..., corresponding to

different roots λ_i, λ_j, ... , where σ_i is regarded as a linear combination with coefficients not all zero of the s_i linearly independent solutions corresponding to the root λ_i. But from (VI) we conclude that σ_i must vanish identically, in as much as σ_i involves λ_i but none of the other λ's. From this contradiction, we obtain the desired result.

EXAMPLE 1. The most important example is the case where L is the operator d/dt and C is the class of entire functions of the complex variable t. Then

$$\varphi_\lambda(t) = e^{\lambda t} \text{ and } g_{p\lambda}(t) = t^p,$$

while R is the entire complex plane. Conditions I, II, III, IV, V, VI are fulfilled. Our system (1) is a system of differential equations with constant coefficients. It is known from other considerations based on the existence theorems for differential equations,[10] that such a system should have a number of linearly independent solutions equal to the degree of $\Delta(\lambda)$. This is exactly the number furnished by the present theory.

EXAMPLE 2. Another important example is the case where L is defined as follows: $Lx(t) = x(t + 1)$. We may take C again as the class of entire functions of t. We have

$$\varphi_\lambda(t) = \lambda^t \text{ and } g_{p\lambda}(t) = \frac{t(t-1)\ \ldots\ (t-p+1)}{\lambda^p}$$

R is the complex plane with the origin deleted.[11] Conditions I to VI inclusive are again fulfilled. The system (1) is now a system of difference equations with constant coefficients.[12] Since R does not include the origin, we have a degenerate situation if $\Delta(0) = 0$. Otherwise our theory gives us a number of linearly independent solutions equal to the degree of $\Delta(\lambda)$.

The special case of a system of difference equations of the type

$$x_i(t+1) = \sum_{j=1}^{m} a_{ij}x_j(t), \qquad i = 1, 2, \ldots, m; \quad \det|a_{ij}| \neq 0,$$

is important enough to invite further consideration: $\Delta(\lambda)$ in this case is $\det|a_{ij} - \delta_{ij}\lambda|$ and is of degree m. The m linearly independent solutions furnished by our theory will depend linearly on functions of the type

$$t^{k_i}\lambda_i^t,$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct roots of $\Delta(\lambda) = 0$, k_i taking on non-negative integral values less than the multiplicity of the accompanying root λ_i. Let the columns of an m square matrix $X(t)$

be the m linearly independent solutions furnished by our theory. If it were possible for det $X(t_0) = 0$ for some particular value of t_0, we could find a linear combination of the columns of X with constant coefficients (not all zero) which would vanish for $t = t_0$. But, since this linear combination of the columns of X is a solution of the system of difference equations, it would have to vanish not only at t_0 but also at $t_0 \pm 1$, $t_0 \pm 2$, $t_0 \pm 3$, ... On account of the special nature of the functions

$$t^{k_i}\lambda_i^t,$$

it would then be necessary for the linear combination of the columns of X to vanish identically,[13] which would be in contradiction with the definition of X. We conclude that det $X(t_0) \neq 0$ for any t_0. It follows that we can build up from our system of m linearly independent solutions a solution which for $t = t_0$ takes on assigned initial values.

If, instead of using constants in forming our linear combinations, we use periodic functions of period 1, we can build up a solution which takes on assigned values on any interval of length 1 closed at one end and open at the other.

FOOTNOTES

1. The literature on systems of differential equations with constant coefficients is enormous. Cf. F. R. Moulton, Differential Equations, New York (1930), pp. 246-294, where a historical review of the contributions of Weierstrass, Thome and Nyswander is included. Cf. also H. Jeffreys, Operational Methods in Mathematical Physics, Cambridge Mathematical Tracts, No. 23 (1927) and F. G. C. Poole, Introduction to the Theory of Linear Differential Equations, Oxford (1936), especially pp. 27-30, and the work of Ernst Snapper cited in the next footnote.

   For the theory of the single linear difference equation, cf. N. E. Nörlund, Vorlesungen ueber Differenzenrechnung, Berlin (1924), pp. 295-300.

2. Cf. Ernst Snapper, Polynomial Matrices in one variable, differential equations and module theory, American Journal of Mathematics, volume 69 (1947), pp. 299-326, especially Sections 1.4-1.6, and Polynomial matrices in several variable, ibid. pp. 622-652, especially Part I.

3. i.e. if $f \epsilon C$ and $g \epsilon C$, then $(af + bg) \epsilon C$, where a and b are any two numbers.

4. i.e. $L(af + bg) = aLf + bLg$.

5. i.e. if $f \epsilon C$, then $Lf \epsilon C$.

6. That these conditions are consistent is proved by the examples in 5. No claim is made that the conditions are independent of each other. Condition I, for example, would hold automatically if C were assumed complete.

7. For differential equations the solution in this form is to be found in the textbook of Frazer, Duncan and Collar, Elementary Matrices, Cambridge (1938), pp. 167-168, but there is no proof that exactly s linearly independent solutions are to be obtained in this way corresponding to each s-fold root λ_1. The filling of this gap was the original purpose of the present paper.

8. J. H. M. Wedderburn, Lectures on Matrices, American Mathematical Society Colloquium Publications, vol. 17 (1934); pp. 33-36.

9. It would be pedantic to worry about the exact definition of $F^{(p-r)}(0)$ or $W^{(p-r)}(0)$ when $r > p$, since they occur in our discussion only when multiplied by zero.

10. Cf. E. L. Ince, Ordinary Differential Equations, London (1927), Chapter 6.

11. Strictly speaking the complex plane should be cut along an arbitrary curve extending from the origin to the point at ∞. $\lambda^t = e^{\mu t}$, where μ is a particular determination of log λ.

12. Cf. N. E. Nörlund, loc. cit.

13. See for example Lemma I of my paper, Formal Power Series Transformations, Duke Mathematical Journal, v. 5 (1939), pp. 794-805, especially p. 795.

The Johns Hopkins University

SOME FORMULAS OF OLTRAMARE

L. Carlitz

1. The following two formulas of Oltramare [5] are reproduced in Dickson's History [2, p. 277].

$$(1.1)\quad 1 + (m!)^4 \equiv -2\left\{\left(\frac{1}{3}\right)^2 + \left(\frac{1.5}{3.7}\right)^2 + \left(\frac{1.5.9}{3.7.11}\right)^2 + \dots\right\} \pmod{4m+1},$$

$$(1.2)\quad 2^5 + (m!)^4 \equiv -2^6\left\{\left(\frac{3}{1}\right)^2 + \left(\frac{3.7}{1.5}\right)^2 + \left(\frac{3.7.11}{1.5.9}\right)^2 + \dots\right\} \pmod{4m+3},$$

where the moduli are primes.

In this note we shall exhibit a number of formulas similar to the above and indicate their connection with generalized hypergeometric series. Throughout the paper p will denote a prime ≥ 3. All series in the paper (except (4.1) below) terminate. As usual, vacuous products = 1.

2. In the first place, if in the formulas

$$(2.1)\qquad \sum_{r=0}^{m}\binom{m}{r} = 2^m, \qquad \sum_{r=0}^{m}\binom{m}{r}^2 = \binom{2m}{m},$$

we take p prime = $2m + 1$, so that $m \equiv -½ \pmod p$, we get after a little manipulation

$$(2.2)\qquad \sum_{r=0}^{m}(-1)^r\,\frac{1.3.5\dots(2r-1)}{2.4.6\dots 2r} \equiv 2^m \equiv (-1)^{\frac{1}{8}(p^2-1)} \pmod p,$$

$$(2.3)\qquad \sum_{r=0}^{m}\left(\frac{1.3.5\dots(2r-1)}{2.4.6\dots 2r}\right)^2 \equiv -\frac{1}{(m!)^2} \equiv (-1)^{\frac{1}{2}(p-1)} \pmod p,$$

where we have made use of Wilson's theorem as well as the theorem on the quadratic character of 2.

In the second place, if in (2.1) we take $p = 4m + 1$, we get

$$(2.4)\qquad \sum_{r=0}^{m}(-1)^r\,\frac{1.5.9\dots(4r-3)}{4.8.12\dots 4r} \equiv 2^m \pmod{p = 4m+1}.$$

$$(2.5)\qquad \sum_{r=0}^{m}\left(\frac{1.5.9\dots(4r-3)}{4.8.12\dots 4r}\right)^2 \equiv \binom{2m}{m} \pmod{p = 4m+1}.$$

(If we make use of the biquadratic character of 2, the right member of (2.4) can be reduced further.) It is clear how additional formulas similar to (2.2), ... , (2.5) can be constructed.

If in place of (2.1) we use the more recondite formula [3]

$$(2.6) \qquad \sum_{r=0}^{am} (-1)^r \binom{2m}{r}^3 = (-1)^m \frac{(3m)!}{(m!)^3},$$

we find by means of Wilson's theorem that

$$(2.7) \qquad \sum_{r=0}^{2m} \left(\frac{1.3.5\ldots(2r-1)}{2\ 4\ 6\ldots 2r}\right)^3 \equiv -\frac{1}{(m!)^4} \quad (\text{mod } p = 4m+1).$$

and

$$(2.8) \qquad \sum_{r=0}^{2m} \left(\frac{3.5.7\ldots(2r+1)}{2.4.6\ldots 2r}\right)^3 \equiv -\frac{16}{5(m!)^4} \quad (\text{mod } p = 4m+3).$$

If we take $p = 3m + 1$ in (2.6), then we get

$$(2.9) \qquad \sum_{r=0}^{2m} \left(\frac{1.4.7\ldots(3r-2)}{3.6.9\ldots 3r}\right)^3 \equiv -\frac{1}{(m!)^3} \quad (\text{mod } p = 3m+1),$$

while if $p = 3m + 2$ we find

$$(2.10) \qquad \sum_{r=0}^{2m} \left(\frac{2.5.8\ldots(3r-1)}{3.6.9\ldots 3r}\right)^3 \equiv -\frac{1}{(m!)^3} \quad (\text{mod } p = 3m+2);$$

in (2.7), ... , (2.10) p is of course assumed prime. It is clear how other formulas of this kind can be obtained.

3. We now make use of a number of formulas that are derived from the Dougall-Ramanujan formula [1], [4]. We shall by no means attempt to construct an exhaustive set of formulas but merely give a few samples.

In the first place the formula [1, p. 96].

$$(3.1) \qquad 1 + 3\frac{m-1}{m+1} + 5\frac{(m-1)(m-2)}{(m+1)(m+2)} - \ldots = m$$

yields

$$(3.2) \qquad 1 - 3\frac{5}{3} + 5\frac{5.9}{3.7} - 7\frac{5.9.13}{3.7.11} + \ldots \equiv -\frac{1}{4} \quad (\text{mod } 4m+1)$$

and

$$(3.3)\qquad 1 - 3\,\frac{3}{5} + 5\,\frac{3.7}{5.9} - 7\,\frac{3.7.11}{5.9.13} + \ldots \equiv \frac{1}{4} \pmod{4m - 1}.$$

Note that (3.1) also implies such congruences as

$$1 - 3\,\frac{7}{1} + 5\,\frac{7.11}{1.5} - 7\,\frac{7.11.15}{1.5.9} + \ldots \equiv -\frac{3}{4} \pmod{4m + 3}.$$

Similar results follow from

$$1 - 3\,\frac{m - 1}{m + 1} + 5\,\frac{(m - 1)(m - 2)}{(m + 1)(m + 2)} - \ldots = 0 \qquad (m \geq 2).$$

If we take

$$(3.4)\qquad 1 - \frac{1}{3}\,\frac{m - 1}{m + 1} + \frac{1}{5}\,\frac{(m - 1)(m - 2)}{(m + 1)(m + 2)} - \ldots = \frac{2^{4m}(m!)^4}{4m((2m)!)^2},$$

then for $p = 4m + 1$, we find

$$(3.5)\qquad \sum_{r=0}^{m-1} \frac{1}{2r + 1}\,\frac{5.9\ldots(4r + 1)}{3.7\ldots(4r - 1)} \equiv (m!)^4 \pmod{p = 4m + 1};$$

if $p = 4m - 1$, we get

$$(3.6)\qquad \sum_{r=0}^{m-1} \frac{1}{2r + 1}\,\frac{3.7\ldots(4r - 1)}{5.9\ldots(4r + 1)} \equiv 16(m!)^4 \pmod{p = 4m - 1}.$$

In the next place

$$(3.7)\qquad 1 + \left(\frac{m - 1}{m + 1}\right)^2 + \left(\frac{(m - 1)(m - 2)}{(m + 1)(m + 2)}\right)^2 + \ldots = \frac{2m}{4m - 1}\,\frac{(m!)^4(4m)!}{((2m)!)^4}$$

yields

$$(3.8)\qquad \sum_{r=0}^{m-1} \left(\frac{5.9\ldots(4r + 1)}{3.7\ldots(4r - 1)}\right)^2 \equiv -\frac{1}{4}(m!)^4 \pmod{p = 4m + 1}$$

and

$$(3.9)\qquad \sum_{r=0}^{m-1} \left(\frac{3.7\ldots(4r - \ )}{5.9\ldots(4r + 1)}\right)^2 \equiv -8(m!)^4 \pmod{p = 4m - 1}.$$

The formula

$$(3.10)\quad 1 - 3\left(\frac{m-1}{m+1}\right)^3 + 5\left(\frac{(m-1)(m-2)}{(m+1)(m+2)}\right)^3 - \cdots = \frac{(m!)^3(3m-2)!}{((2m-1)!)^3}$$

may be applied in exactly the same way. We mention only the following instances:

$$(3.11)\quad \sum_{r=0}^{m-1} (2r+1)\left(\frac{4.7\ldots(3r+1)}{2.5\ldots(3r-1)}\right)^3 \equiv -\frac{4}{27}(m!)^6 \pmod{p = 3m+1},$$

$$(3.12)\quad \sum_{r=0}^{m-1} (2r+1)\left(\frac{2.5\ldots(3r-1)}{4.7\ldots(3r+1)}\right)^3 \equiv -27(m!)^6 \pmod{p = 3m-1}$$

If we take $p = 4m+1$, say, in (3.10) we get a somewhat more complicated result, namely

$$(3.13)\quad \sum_{r=0}^{m-1} (2r+1)\left(\frac{5.9\ldots(4r+1)}{3.7\ldots(4r-1)}\right)^3 \equiv (-1)^{m-1}\frac{2}{21}\frac{(m!)^2}{(2m)!} \pmod{p = 4m+1}.$$

4. The formula of Dougall

$$(4.1)\quad 1 - \frac{2xyz}{(x+1)(y+1)(z+1)} + \frac{2x(x-1)y(y-1)z(z-1)}{(x+1)(x+2)(y+1)(y+2)(z+1)(z+2)} - \cdots = \frac{\Gamma(x+1)\Gamma(y+1)\Gamma(z+1)\Gamma(x+y+z+1)}{\Gamma(y+z+1)\Gamma(z+x+1)\Gamma(x+y+1)}$$

implies many interesting special results. In particular for $x = y = z = m$, (4.1) becomes

$$(4.2)\quad 1 - 2\left(\frac{m}{m+1}\right)^3 + 2\left(\frac{m(m-1)}{(m+1)(m+2)}\right)^3 - \cdots = \frac{(m!)^3(3m)!}{((2m)!)^3}.$$

Then as before, (4.2) implies

$$(4.3)\quad 1 + 2\left\{\left(\frac{1}{2}\right)^3 + \left(\frac{1.4}{2.5}\right)^3 + \left(\frac{1.4.7}{2.5.8}\right)^3 + \ldots\right\} \equiv (m!)^6 \pmod{p = 3m+1}$$

and

$$(4.4)\quad 1+2\left\{\left(\frac{2}{1}\right)^3+\left(\frac{2.5}{1.4}\right)^3+\left(\frac{2.5.8}{1.4.7}\right)^3+\ldots\right\} \equiv -\frac{1}{27}(m!)^6 \pmod{p=3m+2};$$

similarly we get

$$(4.5)\quad 1+2\left\{\left(\frac{1}{3}\right)^3+\left(\frac{1.5}{3.7}\right)^3+\left(\frac{1.5.9}{3.7.11}\right)^3+\ldots\right\} \equiv (-1)^m\frac{(m!)^2}{(2m)!}$$

$$(\bmod\ p = 4m+1)$$

and

$$(4.6)\quad 1+2\left\{\left(\frac{3}{1}\right)^3+\left(\frac{3.7}{1.5}\right)^3+\left(\frac{3.7.11}{1.5.9}\right)^3+\ldots\right\} \equiv (-1)^{m-1}\frac{4(m!)^2}{5(2m)!}$$

$$(\bmod\ p = 4m+3).$$

Again, if following Dougall we take $z = -\frac{1}{2}$, $x = y = m$, in (4.1), we find after a little simplification

$$(4.7)\quad 1+2\left(\frac{m}{m+1}\right)^2+2\left(\frac{m(m-1)}{(m+1)(m+2)}\right)^2+\ldots = \frac{(m!)^4(4m)!}{((2m)!)^4}.$$

Now for $p = 4m+1$, (4.7) yields

$$(4.8)\quad 1+2\left\{\left(\frac{1}{3}\right)^2+\left(\frac{1.5}{3.7}\right)^2+\ldots\right\} \equiv -(m!)^4 \pmod{p=4m+1},$$

while for $p = 4m+3$ we get

$$(4.9)\quad 1+2\left\{\left(\frac{3}{1}\right)^2+\left(\frac{3.7}{1.5}\right)^2+\ldots\right\} \equiv -2^{-5}(m!)^4 \pmod{p=4m+3}.$$

Clearly (4.8) and (4.9) are identical with (1.1) and (1.2), the formulas of Oltramare.

We remark that for $p = 4m-1$, (4.7) implies

$$(4.10)\quad 1+\frac{2}{5^2}\left\{1+\left(\frac{3}{9}\right)^2+\left(\frac{3.7}{9.13}\right)^2+\ldots\right\} \equiv 0 \pmod{p=4m-1},$$

while for $p = 3m+1$, $3m+2$ we find

$$(4.11)\quad 1+2\left\{\left(\frac{1}{2}\right)^2+\left(\frac{1.4}{2.5}\right)^2+\left(\frac{1.4.7}{2.5.8}\right)^2+\ldots\right\} \equiv 0 \pmod{p=3m+1},$$

$$(4.12)\quad 1+2\left\{\left(\frac{2}{1}\right)^2+\left(\frac{2.5}{1.4}\right)^2+\left(\frac{2.5.8}{1.4.7}\right)^2+\ldots\right\}\equiv 0 \quad (\bmod p = 3m+2).$$

On the other hand $p = 5m + 1$ gives

$$(4.13)\quad 1+2\left\{\left(\frac{1}{4}\right)^2+\left(\frac{1.6}{4.9}\right)^2+\left(\frac{1.6.11}{4.9.14}\right)^2+\ldots\right\}\equiv (-1)^{m-1}\frac{(m!)^3}{((2m)!)^4},$$

$$(\bmod p = 5m+1),$$

and so on.

5. Since writing the above the author has obtained a copy of Oltramare's paper. It may be of interest to indicate briefly Oltramare's proof of (1.1) and (1.2). In the second of (2.1) above, replace m by $2m$ and divide both members by the middle term, that is, by

$$\binom{2m}{m}^2.$$

A little manipulation gives

$$1+2\sum_{r=1}^{m}\left(\frac{m(m-1)\ldots(m-r+1)}{(m+1)(m+2)\ldots(m+r)}\right)^2=\frac{(4m)!}{\{(m+1)\ldots(2m)\}^4},$$

which is identical with (4.7). As we have already seen, (1.1) and (1.2) are easily obtained from this formula.

Duke University

REFERENCES

1. W. N. Bailey, *Generalized hypergeometric series*, Cambridge, 1935.

2. L. E. Dickson, *History of the Theory of Numbers*, Washington, 1919, vol. 1.

3. A. C. Dixon, *On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem*, Messenger of Mathematics, vol. 20(1891), pp. 79-80.

4. J. Dougall, *On Vandermonde's theorem and some more general expansions*, Proceedings of the Edinburgh Mathematical Society, vol. 25(1907), pp. 114-132.

5. G. Oltramare, *Note sur la fonction* $G_m = \frac{1.2.3\ldots m}{(m+1)(m+2)\ldots 2m}$, Mem. de l'Institut Nat. Genevois, vol. 4(1856), pp. 33-36.

COLLEGIATE ARTICLES

Graduate training not required for reading

•

DERIVATION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

R. M. Redheffer

Introduction - In many derivations of partial differential equations, the results are simplified from the very first by assuming that some of the variables are small. On the whole, this procedure of linearizing the equations as one goes along is both natural and justifiable. Still, in a few cases it is instructive to do the simplification at the end of the discussion, and such is one of our purposes here. A second purpose is to illustrate the use of the mean value theorem in the setup of incremental relations. For simplicity, the functions and derivatives which occur are assumed continuous.

The present discussion draws heavily on [1] - [8], and should be regarded as expository. Still, some things may be new in detail. For example the equation for sound waves is derived on the assumption that the pressure and density satisfy a general functional relationship $p = f(\rho)$. This is begun in [1], but seems not to have been completed anywhere. Extending to the case in which the equation of state is $T = f(p,v)$, we re-examine certain procedures in elementary thermodynamics that are sometimes presented with a careless use of differentials. Again, in fluid motion the method of approximating only at the end leads to a nonlinear equation for canal waves, which seems not to have been given previously. Similarly we find a possibly new generalization of Bernoulli's theorem, and a few odd features in the theory of transverse vibrations of a bar. These extensions and novelties are not deep, but it is hoped that students and teachers concerned with applied mathematics may find them interesting.

Sound - Suppose given a straight tube of constant cross-section containing a gas initially of density ρ_0 and pressure p_0. Now let a longitudinal wave be propagated so that the layer of molecules initially at position x along the tube moves to a new position $u(x,t)$ at time t, the density and pressure being $\rho(x,t)$ and $p(x,t)$. Then the equation of continuity may be written

$$\rho_0 \Delta x = u(x,t)\Big]_x^{x+\Delta x} \quad \tilde{\rho}(x,t) \tag{1}$$

where $\tilde{\rho}$ stands for an intermediate value of ρ.

Physically, (1) says that the mass of gas in the interval $(x, x+\Delta x)$ is preserved, as this interval expands and contracts under the motion. Mathematically, the existence of $\tilde{\rho}$ with the stated properties follows from the mean value theorem for integrals. Henceforward we introduce such relations without comment. Letting $\Delta x \to 0$ and using the fact that then $\tilde{\rho} \to \rho$, we have

$$\rho_0 = \rho u_x. \tag{2}$$

Similarly Newton's law "$F = ma$" is

$$\Delta x \tilde{\rho} \tilde{u}_{tt} = -p\Big]_x^{x+\Delta x} \tag{3}$$

where $\tilde{u}$ refers to the center of gravity, so that

$$\rho u_{tt} + p_x = 0. \tag{4}$$

If the equation of state is

$$p/p_0 = f(\rho/\rho_0) \tag{5}$$

then (2), (4), and (5) combine to give

$$u_x u_{tt} = (p_0/\rho_0) f'(\rho/\rho_0) u_{xx} \tag{6}$$

after simplification.

We define waves of small amplitude to be those for which $\rho \cong \rho_0$. Since (2) then implies $u_x \cong 1$, Eq. (6) simplifies to give the following:

Suppose a medium satisfies (5), *where $f'(s)$ is continuous near $s = 1$. Then it propagates waves of small amplitude if and only if $f'(1) > 0$, and the velocity is*

$$\sqrt{(p_0/\rho_0) f'(1)} \;.$$

Thermodynamics - For a gas the equation of state is generally taken as

$$T = F(p,v) \tag{7}$$

rather than (5), where T is temperature and v volume. To proceed from (7) to (5), note first that $\Delta U = \Delta Q - \Delta W$, where U is internal

energy, W is work, and Q is heat energy. This is merely the principle of conservation of energy. - It is not hard to set up an inequality for the work ΔW, which gives $\Delta W = \bar{p}\Delta v$ by continuity. Hence

$$\Delta Q = M\Delta p + N\Delta v + o(\Delta v) \tag{8}$$

where M and N are functions of p and v. (It is misleading to write $dQ = \ldots$ as in [1] and [2], since dQ depends on the path followed; thus "dQ" is not really the differential of anything).

From (7) one has

$$\Delta T = F_p\Delta p + F_v\Delta v + o(\Delta v), \tag{9}$$

if F is differentiable, and this combines with (8) to give us the information we need later.

The heat capacity c is defined as $\lim \Delta T/\Delta Q$. Since the limit depends on the path followed, it is customary to distinguish the two cases c_p for constant pressure, and c_v for constant volume. Setting $\Delta p = 0$ in (8), (9), dividing, and taking the limit, gives $c_p = F_v/N$, and similarly for c_v. Thus M and N in (8) are known.

Because of the speed with which they occur, the changes of state in a gas propagating sound are adiabatic; no heat is lost, and $\Delta Q = 0$. In this case (8) gives $dp/dv = -N/M$. Now at last we have a hold on f', the derivative of the function in (5). For, considering a unit mass of gas gives $\rho v = 1$, so that (5) is $p/p_0 = f(v_0/v)$; and one may differentiate with respect to v.

Putting these remarks together gives the following:

The velocity of propagation for sound waves of small amplitude is

$$\sqrt{c_pF_v/c_vF_p}\ \rho_0^{-1}$$

where c_p and c_v are the specific heats at constant pressure and volume, the equation of state is $T = F(p,v)$, and the quantities are evaluated at p_0, ρ_0. In particular the velocity is

$$\sqrt{p_0c_p/\rho_0c_v},$$

as it should be, when (7) is $pv = nRT$.

Canal waves - The use of absolute position as above, rather than relative displacement, also leads to a simple interpretation of canal waves [4], [5], [6]. We give only a brief outline. Let $u(x,y,t)$ and $v(x,y,t)$ be the coordinates at time t of a fluid particle initially at (x,y), so that *the motion gives a family of mappings of the (x,y) plane onto the (u,v) plane.* The equation of continuity

takes the form $u_x v_y = 1$, provided the density is constant. This combines with conservation of momentum to give $u_x u_{tt} = -g v_{ox}$ where, as in [5], the subscript o refers to the surface. (our $\tilde{}$ notation is useful here).

If the wavelength is much greater than the depth, then vertical planes remain vertical, so that $v = y/u_x$ by integration as in [5]. The equation of motion is now

$$(u_x)^3 u_{tt} = ghu_{xx} \tag{10}$$

which becomes the well-known result when $u_x \cong 1$; and this entails only $y \cong v$. Thus it entails only that the waves have small amplitude. Deep-water waves may be similarly treated.

Streamline flow - If s represents distance along a tube of flow and $A = A(s)$ the cross-section area, then conservation of mass leads to

$$(d/dt)\ \widetilde{\widetilde{(\rho A \Delta s)}} = 0 \tag{11}$$

by inspection, and conservation of momentum gives

$$(d/dt)\ (\widetilde{\widetilde{\rho A \Delta s}}\tilde{v}) = -pA\Big]_s^{s+\Delta s} + A\Big]_s^{s+\Delta s}\tilde{p} + F\rho(\Delta s)A \tag{12}$$

where v is velocity and F body-force per unit mass, both along s. The second term on the right arises from pressure on the sides of the element of the tube of flow. - Equation (12) combines with (11) to give

$$dv/dt = -p_s + F\rho \tag{13}$$

after letting $\Delta s \to 0$ and using Leibniz' rule twice.

The left side is $\rho(vv_s + v_t)$. Assuming steady-state flow we have $v_t = 0$, so that (13) may be integrated:

$$v^2/2 + \int_{s_0}^{s} (p_s/\rho)\,ds = \int_{s_0}^{s} F\,ds. \tag{14}$$

Here[1] *we have the analogue of Bernoulli's theorem for compressible fluids and nonconservative body forces.* If the fluid is incompressible, (14) reduces to

$$v^2/2 + p/\rho = \int_{s_0}^{s} F\,ds \tag{15}$$

[1]*The derivation of (14) was carried out jointly with P. G. Hodge, Jr.*

and this in turn becomes the standard result when there is a potential Ω, $F = -\Omega_s$. Note that *the integrals in (14) are well-defined, since they are taken along the tube of flow.*

Stiff bar - If a straight bar is bent so that its "neutral fiber" follows the curve $y(x,t)$ at time t, then the bending moment at cross section x will be $EI\phi_x$, where $\tan\phi = y_x$ and I is the moment of inertia of the cross section about a suitable axis. Except for terms of order y_x^2 this reduces to EIy_{xx}, of course, but we want to keep the more exact form at first. Following through the usual treatment [1] - [8] but keeping the error terms we find

$$(\partial/\partial x)EI\phi_x + F = \rho I\phi_{tt}, \quad F_x = \rho A y_{tt}, \quad \tan\phi = y_x \tag{16}$$

for transverse vibrations of a bar of density ρ and cross-section area A. The solutions of this system are among those of the single equation

$$(\partial^2/\partial x^2)EI\phi_x + \rho A y_{tt} = (\partial/\partial x)\rho I\phi_{tt}$$

which becomes

$$EI\phi_{xxx} + \rho A y_{tt} = \rho I\phi_{xtt}$$

when E, I, ρ, A are constant. If we could replace ϕ by y_x the latter would be the form usually given,

$$EIy_{xxxx} + \rho A y_{tt} = \rho I y_{xxtt}. \tag{17}$$

Unfortunately ϕ is not well enough approximated by y_x to permit the three differentiations necessary: even the leading term is nonlinear. Specifically, one gets

$$EIY_{xxxx} + \rho A y_{tt} = \rho I(y_{xxtt} - 2y_{xx}y_{xt}^2) \tag{18}$$

which is the same as (17) only when y_{xx} is small.

Physically the assumption of small curvature is surely justified, but we are concerned now with a problem in pure mathematics. It will be seen that *the solutions of (18) which are relevant to the problem are all contained among those of (17) even when the equations themselves differ significantly.*

Small y_x (and constant ρ, I, A) really does allow us to write

$$EIy_{xx} + F = \rho I y_{xxt}, \quad F_x = \rho A y_{tt} \tag{19}$$

in place of (16). Eliminating F by differentiation we get precisely (17), so that the italicized assertion is proved: The solutions desired are among those of (19), and these in turn are among those of (17). (We are not concerned here with the deep question of whether the solutions are well approximated when the equation is. All we ask is that the equation itself be correct up to the leading terms in the quantity assumed small).

Thus *it is by no means always best to make the approximations at the very end;* a better and equally correct result can sometimes be found by approximating at a suitably chosen point near the beginning. The author finds these results especially curious in that there seems to be no way of going directly from (18) to (17) on the sole assumption that y_x is small.

REFERENCES

1. Rayleigh, The Theory of Sound, Dover Publications, (1945) p 18-23, 30-32.

2. Morse, Vibration and Sound, Macmillan, (1948) p 218-222.

3. Webster, Partial Differential Equations of Mathematical Physics, Hafner Publishing Co. (1947) p 27-30.

4. Milne-Thompson, Theoretical Hydrodynamics, Macmillan, (1949) Ch. XIV.

5. Page, Introduction to Theoretical Physics, Van Nostrand Co., (1935) p 241-243.

6. Bateman, Partial Differential Equations of Mathematical Physics, Dover Publications, (1944) p 67-71.

7. Lamb, Hydrodynamics, Dover Publications, (1945) p 254-256.

8. Timoshenko, Vibration Problems in Engineering, Van Nostrand, (1937) p 331-338.

University of California
Los Angeles

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

IS THE ALGEBRA TAUGHT IN COLLEGES REALLY 'COLLEGE ALGEBRA'?

Margaret F. Willerding

After a most discouraging fifty minute period teaching a College Algebra class, I began to wonder; "Is the algebra taught in college really 'College Algebra'?" In seeking an answer to my question, I read the descriptions of the freshman algebra courses offered at Harris Teachers and Junior College in St. Louis.

There are three freshman courses of algebraic content described in the catalogue. The first is a combination course in college algebra and trigonometry for pre-engineering students who have had at least one and one-half units of high school algebra and one unit of plane geometry. The algebraic content of this course is:

> Review of elementary algebra; constants and variables; functions and graphs; special products and factoring; operations with simple and complex fractions; exponents; radicals; linear equations; fractional equations; determinants; quadratic and equations of higher degree; j-operator; ratio and proportion; the exponential function; logarithms and the logarithmic function.

I have no quarrel with this course either in theory or in practice. It was the other two courses that bothered me. The first of these is the traditional three hour course called 'College Algebra'. The prerequisites are one unit of high school algebra and one unit of plane geometry. Its content is:

> Review of elementary algebra; radicals; exponents; use of logarithms; fractions; fractional equations; linear

> equations in one and two unknowns; simple determinants; systems of equations; quadratic equations; imaginary and complex numbers; graphs; progressions; ratio; proportion and variation; variables and functions; permutations; combinations; simple probability; mathematical induction; binomial theorem; theory of equations.

These topics are certainly the traditional topics of "College Algebra". But were these topics taught in the college algebra course? In actual practice so much time has to be spent on review of elementary algebra - the fundamentals - that many of the topics listed in the catalogue are omitted.

Although the actual content of the course is more accurately described as intermediate rather than college algebra, the teaching is on a college level.

At all times the postulational approach is used. The postulates being suggested by the students' past experiences in arithmetic. The stress is on ideas rather than on techniques. Although the stress is on ideas, the course is taught in such a way as to provide the needed training in the essential techniques.

Rigor and understanding are each part of mathematical training. As students understand, techniques are improved. It is essential to realize that there is no such thing as absolute rigor. Rigor is a relative thing. It is relative to an individual's mental maturity. Even the good student in his early study of mathematics is usually not sufficiently mature in his thinking to appreciate much mathematical rigor.

With students as mathematically immature as those entering the college algebra course described above, the thing to remember is to teach nothing that will have to be unlearned later.

Much of the material in this college algebra course is a repetition of the topics of high school algebra. This repetition is absolutely essential if the students are to have the needed training in the techniques necessary in the further study of mathematics and its applications. In teaching this repetitious material the stress should be on understanding rather than drill.

As I look backward over past classes I find that each year I teach less and less 'college algebra' and year after year the course becomes more and more college algebra in name only.

But if the traditional college algebra course is no longer college algebra, just what is the algebra taught in Elementary College Mathematics?

The prerequisites of this course is one year of high school mathematics. The algebraic content is:

> Fundamentals of Elementary algebra: symbols, signed numbers, operations with algebraic quantities, formulas, exponents, radicals, special products and factoring. Equations; linear equations in one and two unknowns, simple quadratics in one unknown. Graphs and the function concept.

Most of the students entering this course have had only one unit of high school mathematics. In the majority of cases this unit was General Mathematics. Except for the more advanced viewpoint the algebra taught in Elementary College Mathematics is *elementary algebra.*

Teaching is very important in this course. Many of the students in such a course are "soured" on mathematics. Many of them feel that mathematics is no more than a lot of fancy tricks and techniques. Teaching of the laws of algebra to such students should be a challenge to a mathematics instructor. With the proper approach, he can show the students that mathematics has more to offer a college student than the memorization of formulas and the manipulation of symbols.

I present these facts without apology either for the students at Harris Teachers College or for the algebra courses offered there. At the 1953 meeting of the Missouri Section of the Mathematical Association of America I had occasion to discuss this matter with instructors from most of the colleges and universities in Missouri. The majority of the instructors freely admitted that their schools were faced with the same problem. If students are taught at the level in which they enter the college algebra courses, then the algebra taught in colleges ceases to be college algebra.

Harris Teachers College,
St. Louis, Mo.

EDITOR'S COMMENT: Professor Willerding is a brave soul to confess this in public. By implication she is echoing the oft-heard complaint that our entering college students (freshmen) are poorly, if at all, prepared even in elementary algebra. Apparently at Harris Teachers College they meet the situation by really (re)teaching elementary algebra. That certainly is one solution. But why not call the course "Elementary Algebra from a Near Mature Viewpoint", or some such; rather than College Algebra which it obviously is not?

Is the situation that Professor Willerding describes regional? If widespread, what do others do about it? Let us hear from our readers.

THE PARABOLA OF SURETY

R. F. Graesser

Galileo discovered that the trajectory or path of a projectile *in vacuno* is a parabola. The envelope of all parabolic trajectories obtained by varying the angle of elevation of the gun is another parabola, called the *parabola of surety* because no point above it can be reached by a projectile from the gun. It is the locus of the highest points attainable by the gun. What follows is a derivation of the parabola of surety by the methods of analytic geometry.

The subject of envelopes is often not studied until the first course in differential equations. However, the equation of the parabola of surety may be readily obtained by analytic geometry. To this end, let the gun be located in a vertical xy-plane with its muzzle at the origin. Let θ be its angle of elevation measured from the positive end of the horizontal x- axis. Then the equation of the trajectory *in vacuo* is

$$y = x \tan\theta - (g\, x^2 \sec^2\theta)/2v_0^2,$$

where v_0 is the muzzle velocity, and g is the acceleration of gravity. also

$$y = -(g\, x^2 \tan^2\theta)/2v_0^2 + x\tan\theta - gx^2/2v_0^2, \qquad (1)$$

if we replace $\sec^2\theta$ by $1 + \tan^2\theta$. To obtain the maximum height attainable by the gun at a given horizontal distance x from the muzzle, we need to hold x fixed and maximize y in (1) as a function of θ and, therefore, as a function of $\tan\theta$. From analytic geometry the vertex of the parabola

$$y = ax^2 + b\,x + c$$

is $[-b/2a,\ (4ac - b^2)/4a]$.
In other words, $(4ac - b^2)/4c$ is the maximum value of the quadratic function in the right member. Applying this to (1) as a quadratic function of $\tan\theta$, we have the maximum value of y for a given x as

$$y = -gx^2/2v_0^2 + v_0^2/2g.$$

If now x is allowed to vary, this becomes the equation of the locus of highest points attainable by the gun or the equation of the parabola of surety. Curiously this parabola has its focus at the origin or at the muzzle of the gun. Its positive x-intercept is v_0^2/g, which is the maximum range of the gun. The coordinates $(v_0^2/g,\ 0)$ of a point at maximum range must satisfy (1) when θ is the angle of maximum range. Substituting these coordinates in (1) we have the equation

$$(\tan\theta - 1)^2 = 0$$

for obtaining θ as the angle of maximum range. Hence this angle is $45°$.

University of Arizona

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

AN OLD TIME COMPUTER

William R. Ranson

A device often brought to the attention of puzzled teachers of mathematics consists of two flat pieces of bone or ivory (see Fig. 1) hinged at one end and ruled with divergent scales marked *L, S, T, C*, etc., and sometimes with three additional scales parallel to its long sides and marked *N, S, T*.

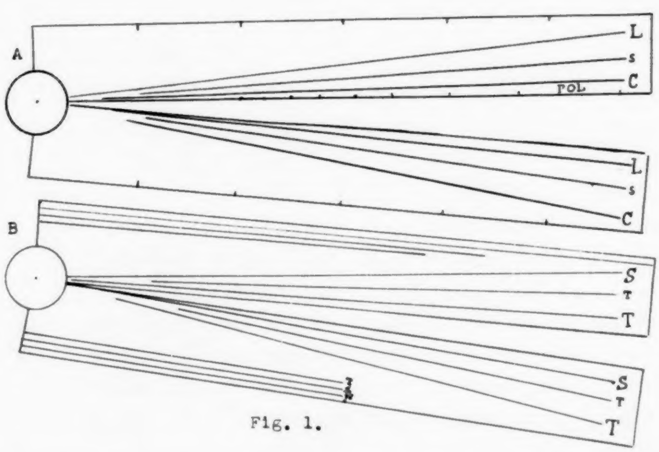


Fig. 1.

In the first edition of Bowditch's "Navigator" (1802) there is a chapter called "The Sector" which explains this device. With the aid of a pair of dividers, it enabled our forebears to do proportion, solve right triangles, lay off angles, and construct regular polygons. The earliest description we have seen is in the second edition of Edmond Gunter's "Description and Use of the Sector, Crosse Staffe, and Other Instruments" (1636), which followed a first edition about a decade after Gunter's death.

When the sector is opened out straight, the upper edge of side *A* gives a foot long scale of inches.

The scales that converge at the center of the hinge are used with the sector opened the proper amount, as we shall explain: similar isosceles triangles then serve to solve a variety of problems by proportion. The fundamental scale for this purpose is on side A, and is marked "L": from this dividers take off "lines", or as we should say, *distances*. L is a uniform scale and is graduated from 1 to 10.

For a simple proportion, $a:b :: c:x$, the two L scales on side A are used. Open the sector so that the distance a spans the opening between the b's on the L scales: then the distance between the c's is the required x. See Fig. 2.

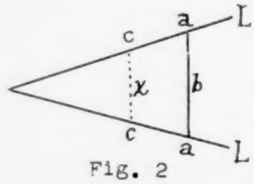


Fig. 2

To find the product $M\cdot\sin\beta$, use the S scales on the B side. Open the sector so that M spans from the right end of one S scale to the right end of the other. Then the span from one β to the other is the required product.

Similarly, using the T scales on side B, $M\cdot\tan\beta$ may be found. The T scales go up to $45°$ only: for angles from $45°$ up to $75°$, the scales marked with a smaller T are used. These scales give the tangents a quarter of their true size, so in using them $4M$ instead of M must span their ends to give $M\cdot\tan\beta$. The s scales are likewise used with $4M$, for the same reason, to find a product $M\cdot\sec\beta$.

The "chord scales", marked "C", are used in laying off angles. A circle is drawn with the vertex of the angle as center. Then its radius is used to span from $60°$ on one C scale to $60°$ on the other. Then the span between the two β's is the chord for the angle β. This scale is constructed by placing β at a distance $2\sin(\beta/2)$ from the hinge. See Fig. 3.

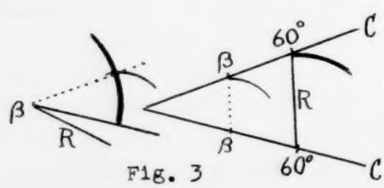


Fig. 3

One other scale, marked "*pol*" gives the chord for the side of a regular inscribed polygon of 4 to 12 sides. The sector is opened so that the distance between the 6's is the radius of the circumscribed circle: then the side of the N-sided regular inscribed polygon is the distance between the N's on the two polygon scales.

The three scales parallel to the long edge, marked *T, S, N*, are logarithmic scales, like those on a modern slide rule. Gunter showed how to use these with the aid of a pair of dividers. The scale marked "*N*" is a two-dekad scale of logarithms, like our *A* and *B* scales, and the other two scales give log-sines and log-tangents.

Tufts College

ON RULED AND DEVELOPABLE SURFACES OF REVOLUTION

Murray S. Klamkin

In this note all the ruled and developable surfaces of revolution will be determined. Since the class of developable surfaces is a subclass of the class of ruled surfaces, one might expect offhand that the class of ruled surfaces of revolution is considerably greater than the class of developable surfaces of revolution. However, it will be shown that the two classes consist of three and two, types of surfaces, respectively.

The parametric equations for ruled surfaces are given by

$$(1) \qquad \begin{aligned} x &= a(t)z + m(t), \\ y &= b(t)z + n(t), \quad (1) \ . \end{aligned}$$

The equation for surfaces of revolution is

$$(2) \qquad X^2 + Y^2 = \phi(z) \ .$$

Substituting *x* and y from (1) and (2):

$$(3) \quad \phi(z) \equiv [a(t)^2+b(t)^2]z^2 + [a(t)m(t)+b(t)n(t)]2z + [m(t)^2+n(t)^2] \ .$$

Thus,

$$(4) \qquad \begin{aligned} a(t)^2 + b(t)^2 &= \text{constant}, \\ a(t)m(t) + b(t)n(t) &= \text{constant}, \\ m(t)^2 + n(t)^2 &= \text{constant}, \end{aligned}$$

Whence,

$$(5) \qquad \phi(z) \equiv Az^2 + Bz + C = x^2 + y^2 \ .$$

Consequently, the only ruled surfaces of revolution are the hyperboloid of one sheet, the right circular cylinder, and the right circular cone. Also, the latter two are the only developable surfaces of revolution, since the class of developable surfaces is a subclass of the class of ruled surfaces, and the hyperboloid is not developable.

A direct method of obtaining all the developable surfaces of revolution is as follows:

The partial differential equation of developable surfaces is given by

$$\left[\frac{\partial^2 z}{\partial x \partial y}\right]^2 - \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = 0, \quad [1]. \tag{6}$$

Substituting

$$z = F(x^2 + y^2) \quad \text{into (6):}$$

$$F^1(r)\,[2rF^{11}(r) + F^1(r)] = 0\ , \tag{7}$$

where

$$r = x^2 + y^2\ .$$

Thus,

$$F(r) = A\sqrt{r} + B\ ,$$

and

$$Z = A\sqrt{x^2+y^2} + B \tag{8}$$

For $A=0$, (8) is the equation of a right circular cylinder. For $A\neq 0$, (8) is the equation of a right circular cone.

References

Goursat-Hedrick, Mathematical Analysis, Vol. 1, pp. 464, 526.

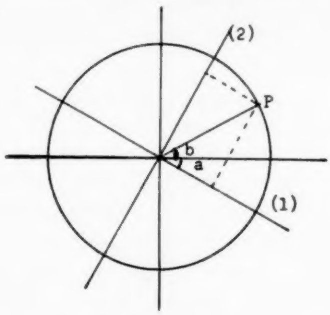
Polythechnic Institute of Brooklyn

SIN $(A + B)$

F. H. Young

In most trigonometry texts the derivation of $\sin(a + b)$, etc., is either lacking in generality or lacking in simplicity. The following derivation is suggested as a means of avoiding both Scylla and Charybdis.

First, let us suppose that the sine and cosine functions have been defined in terms of ordinate and abscissa in the unit circle. Then, let us assume that the normal form of the straight line has been developed as an application of these functions. We are now prepared to expand $\sin(a + b)$. As in the figure, let the angle b be drawn positively and a negatively in the unit circle. Point P, the terminus of angle b, then, has coordinates $(\cos b, \sin b)$. Let line *(1)* be the line coinciding with the terminal side



of angle a. The equation of *(1)*, in normal form, is

$$x \sin a + y \cos a = 0.$$

The distance from P to *(1)* is

$$d = \sin(a + b) = \pm (\sin a \cos b + \cos a \sin b).$$

Since this is to be an identity for all a and b, a consideration of the result when $a = 0^\circ$ shows that the $+$ sign should be used.

Similarly, the equation of line *(2)*, normal to *(1)*, is

$$x \cos a - y \sin a = 0.$$

The distance from P to *(2)* is then

$$d = \cos(a + b) = \cos a \cos b - \sin a \sin b,$$

with the sign determined as before.

If angle a were drawn in a positive direction, the same technique would yield the expansions of $\sin(b - a)$ and $\cos(b - a)$.

Clearly, no restriction is placed on either the magnitude or the sign of a or b.

Montana State College

INTERNATIONAL CONGRESS OF MATHEMATICS, 1954

GENERAL

Date and Place. The Congress will be held in Amsterdam from Thursday September 2nd to Thursday September 9th (inclusive) and will meet in the building of the Royal Tropical Institute. The opening and closing sessions will be held in the Amsterdam "Concertgebouw".

Secretariat. All correspondence must be directed to *The Secretariat of the International Congress of Mathematicians 1954, 2e Boerhaavestraat 49, Amsterdam, The Netherlands.*

SCIENTIFIC PROGRAM

One-hour-addresses: Until now a score of outstanding mathematicians have been invited by the Organizing Committee to deliver one-hour-addresses; most of them have accepted the invitation. The Organizing Committee is convinced that by these addresses a survey of the recent development in the whole field of mathematics will be furnished.

Sections. As stated in the first communication there will be seven sections.

Half-hour-addresses: Most of the 45 experts who, up to the present, have been invited to deliver half-hour-addresses have accepted the invitation. The distribution over the Sections is intended as follows:

Section	I	Algebra and Theory of Numbers	(8 addresses)
"	II	Analysis	(13 ")
"	III	Geometry and Topology	(8 ")
"	IV	Probability and Statistics	(4 ")
"	V	Mathematical Physics and Applied Mathematics	(6 ")
"	VI	Logic and Foundations	(3 ")
"	VII	Philosophy, History and Education	(3 ")

Short lectures: Short lectures will be given by regular members of the Congress who apply beforehand to the Organizing Committee. The time allotted for a short lecture is 15 minutes. The Organizing Committee has the intention to give copies of the collected preprints of these lectures to all regular members at the beginning of the Congress. It is therefore essential that the Organizing Committee

should be in possession of the abstracts of the papers concerned before February 15th, 1954; Address: *Secretariat of the International Congress 1954, 2e Boerhaavestraat 49, Amsterdam, The Netherlands.* These abstracts should be made on the blank enclosed herewith; the text will be considered final. The abstract may not exceed 400 words. It is by no means sure that applications which reach the Organizing Committee after February 15th, 1954 can be accepted.

The California Conference for Teachers of Mathematics is holding its fourth annual meeting on the Los Angeles Campus of the University of California during the period July 6-16, 1954. The Conference is sponsored by the University in co-operation with the California Mathematics Council. General sessions include a wide variety of lectures, panel discussions, and campus tours. The choice of study groups will satisfy a wide range of individual interests. Of special interest are the laboratory groups in elementary and sécondary mathematics where teachers may actually learn to make many of the teaching aids which are so necessary in our modern schools. Two units of credit may be earned by those participating in the Conference. A moderate fee is charged. For further information write to Clifford Bell, Mathematics Extension, University of California, Low Angeles 24, California.

BUT IT SOMETIMES HELPS

Du Pont does not believe in using psychiatry to screen out intensely tense employees or job seekers. The most valuable people are sometimes "screwballs." Says Dr. Dershner: "If you do a good enough screening job you may get people who are perfectly normal. But you will have screened out people who discover things like Nylon."

BUSINESS WEEK

Errata in "Foundations of Operator Mathematics". Vol. 25, No. 5, p. 25

The fifth paragraph of page 251, second line, should read "which *Aa* and *Ba* are defined". The third paragraph of page 252, ninth line, should read "of multiplication by that variable or constant. For example, *nAb*". Page 255, second paragraph, sixth line should read, "of the operand-variable will be called functors. Examples of non-linear". Page 257, the line following equation 2.15 should read, "we must bear in mind that ${}_h\Delta_x F$ means etc.". The equation immediately following equation 3.7, page 258 should read

$$L_h \; {}_h\delta_x^{-1} \; s \; {}_o\delta_x^{-1} \; .$$

Page 261 - The line following equation 4.10 should read "then the operator etc." the *t* is not capitalized. This mistake is of little importance. Equation 4.11 of the same page should read

$$A \overset{s}{=} \sum_{i=o}^{\infty} \frac{(x-x_o)^i \; {}_{x_o}D^i \cdot A}{i\,!}$$

The mistake was that the dot was missing in my notation.

SOME INTRODUCTORY COMMENTS ON FERMAT'S LAST THEOREM*

About 325 years ago, Fermat wrote on the margin of one of his books the statement that

(I) $\qquad x^n + y^n = z^n$, x, y, z and n integers and $n > 3$,

has no solution. He added that he had a beautiful proof but that there was not room on the margin for it. If he had a proof, it is fortunate that there was not room for it; because a great deal of valuable mathematics and mathematical training have come out of seeking for one. Many of our greatest mathematicians and numerous untrained workers have tried to prove this theorem. The writer suspects that if Fermat had a proof it was in what would now be called the elementary field. For a present-day sophomore must know much more mathematics then did Fermat.

This note will present, with brief proofs, suggestions of proofs, or no proofs at all, some of the first observations almost anyone would make about equation (I). As is usually done, we assume that x, y, z satisfy (I) and seek to find a contradiction.

Relative sizes of x, y, and z. There is no loss in generality in assuming that these parameters are positive. Then x and y cannot be equal for if they are $z^n = 2y^n$ and $z = \sqrt[n]{2}\, y$.

Hence we can assume

(1) $$z > y > x.$$

If we substitute $y + (z - y)$ for z in (I), we obtain

$$x^n + y^n = y^n + ny^{n-1}(z - y) + \dots\dots\dots\dots$$

whence

(2) $$z - y < \frac{x^n}{ny^{N-1}} < \frac{x}{n}$$

and

(3) $$0 < 1 - \frac{y}{z} < \frac{1}{n}\frac{x}{z} < \frac{1}{n}$$

It follows that

(4) $$\lim_{n=\infty} \frac{y}{z} = 1$$

**See announcement following "CONTENTS" in Vol. 27, No. 2, Nov.-Dec., 1953. This announcement has been very favorably received.*

One sees in (4) a suggestion that something might be learned about (I) for very large values of n, by studying

$$\lim_{n=\infty} \frac{x}{z}$$

when x, y and z are integers.

The inequality, $z < x + y$, gotten from (2) can be generalized into

$$(3) \qquad \begin{aligned} z^m &< x^m + y^m, \quad m < n \\ z^m &> x^m + y^m, \quad m > n \end{aligned}$$

For if we divide (I) by z we see that

$$z^{N-1} = \frac{x^n}{z} + \frac{y^n}{z} < \frac{x^n}{x} + \frac{y^n}{y} = x^{N-1} + y^{N-1},$$

Similarly

$$z^{n+1} > x^{n+1} + y^{n+1}$$

Simple inductions give (3). These inequalities not only form a basis for comparison of x, y and z, but dispose of the too often made assumption that (I) might have the same solution for different values of y.

We may as well consider x, y *and* z *relatively prime* for if any two contain a common factor, the other one must contain that factor, so its nth power can be divided out of (I).

Obviously *two of these parameters must be odd and one even.*

If the odd ones are x and y and they are of the forms $2m_1 + 1$ and $2m_2 + 1$, then either m_1 or m_2 is odd and the other is even. This and other interesting properties can be found by actual substitution in (I). The situation is different if z is odd.

It suffices to assume that n *is prime*, for if $n = pm$, (I) becomes

$$(x^p)^m + (y^p)^m = (z^p)^m,$$

Since the theorem has been proved for $n = 4$, it is true for even powers which contain 4. (Mr. Grisell has recently proved it for $2(8m + 1)$ provided $8m + 1$ is prime and does not divide x, y, or z. See this magazine Vol. 26, No. 1, Sept.-Oct., 1953, p 263. We, also, have on file a paper that extends Mr. Grisell's results).

The parameters, x, y, z, *are composite* for, since n is odd, the numbers $x^n + y^n$, $z^n - x^n$ and $z^n - y^n$ are factorable, and as will be seen from (12) $z - y$, the least of these, is greater than 1.

The numbers, $x + y$, $z - x$ and $z - y$ are nth powers of integers, in the so-called "first case" that is when x, y, z are prime to n. Because we can write, for instance,

$$(5) \quad [(x+y) - y]^n + y^n = (x+y)\,[(x+y)^{N-1} - n(x+y)^{N-1}y + \ldots - ny^{N-1}]$$

And it follows from the fact that x, y and z are relatively prime and prime to n that the two factors on the right have no factor in common. Hence

$$(6) \quad \begin{aligned} x + y &= u_1{}^n \\ z - x &= u_2{}^n \\ z - y &= u_3{}^n \end{aligned}$$

Also

$$(7) \quad \begin{aligned} x^{N-1} - x^{N-2}\,y + \ldots + y^{N-1} &= v_1{}^n \\ z^{N-1} + z^{N-2}\,x + \ldots + x^{N-1} &= v_2{}^n \\ z^{N-1} + z^{N-2}\,y + \ldots + y^{N-1} &= v_3{}^n \end{aligned}$$

Some important relations result from applying Fermat's Theorem, $t^n \equiv t \bmod n$, to (I) and (6).

From (I) we obtain

$$x + y - z \equiv o \bmod n$$

that is

$$(8) \quad x + y - z = cn$$

The constant, c, is divisable by

$$(x + y)^{\frac{1}{N}},\ (z - x)^{\frac{1}{N}},\ \text{and}\ (z - y)^{\frac{1}{N}}$$

as can be seen from (5). It is obvious that c is even and positive. It has been shown that c contains n. (Incidently, cubing (8) provides a simple proof for the case $n = 3$.

From (6) we obtain

$$(9) \quad \begin{aligned} x + y - (x + y)^{\frac{1}{N}} &= k_1{}^n \\ z - x - (z - x)^{\frac{1}{N}} &= k_2{}^n \\ z - y - (z - y)^{\frac{1}{N}} &= k_3{}^n \end{aligned}$$

Subtracting the first of these identities from the sum of the other two and substituting from (8) gives, after a change of signs,

$$(10) \qquad (z - x)^{\frac{1}{N}} + (z - y)^{\frac{1}{N}} - (x + y)^{\frac{1}{N}} = k\,n\,,$$

where k is readily seen to be even and can be shown to be positive, non-zero, and to contain n. From (10) and (2) we obtain a lower-limit for the least of x, y, z, namely

$$(11) \qquad x < n\,(2\,n^2 + 1)^n$$

Since this "first case" has been proved for $n < 253,\ 547,\ 889$ it follows from (11) that it is true for

$$x < 253,\ 547,\ 889\,[2(253,\ 547,\ 889)^2 + 1]^{253,\ 547,\ 889}$$

Fancifully, if this number were written on a tape 1/4 inch wide and 1/1000 inch thick and this tape were wound around the Earth as a core it would include in the ball all the planets of our planatational system, when only a tiny part of it had been used. This number has been increased. What I would like for some one to do is to prove that it is unlimited.

For further consideration of the approach from this viewpoint, see: Am. Math. Monthly, Vol. XIV, No. 7, p. 441.

However most of the work done in trying to prove Fermat's Last Theorem has been directed toward limiting n. In this connection, see: Math'l Monographs, Diophantine Analysis, Carmichael, p. 85; Vandiver, Am. Math. Monthly, Vol. 53, 10, p. 555, bibliography p. 577; and Dickson's 'immortal' History of the Theory of Numbers, Vol. II, p. 731. This is an all inclusive report up to 1920.

The more important points in the above comments will be found in these references.

But if you have a "hunch" of your own, it might be well to exploit it somewhat before becoming suppressed by the enormity of the work that has been done on this subject.

Address additions or criticisms of these comments to: *Glenn James, 14068 Van Nuys Blvd., Pacoima, California.*

Errata in "Proof of Fermat's Last Theorem for $n = 2(8a + 1)$". Vol. 26, No 5, p. 263.

In line 12, replace $x^2 - y^2$ by $z^2 - y^2$.

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter cuestions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

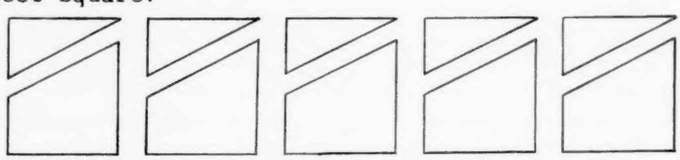
Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles, 29, California.

PROPOSALS

194. *Proposed by C. S. Ogilvy, Hamilton College, New York.*

Rearrange the five quadrilaterals and five triangles to form one perfect square:



195. *Proposed by Leo Moser, University of Alberta.*

If $[x]$ denotes the greatest integer not exceeding x show that:

$$\left[\frac{1}{\pi^2/6 - \sum_{i=1}^{n} 1/i^2} \right] = n .$$

196. *Proposed by G. W. Courter, Baton Rouge, Louisana.*

Construct a right triangle given its hypotenuse and a point on it which is the corner of the inscribed square on the hypotenuse.

197. *Proposed by A. S. Gregory, University of Illinois.*

Let the explicit expression for the nth term of a sequence $\{K_n\}$ be known. Find an explicit expression for the nth term of a sequence $\{\varphi_n\}$ which is defined as follows:

$$\varphi_n = \varphi_{n-1} + \varphi_{n-2} + K_n, \quad n = 2, 3, \ldots\ldots$$

with φ_0 and φ_1 given

198. *Proposed by C. W. Trigg, Los Angeles City College.*

A marble of radius r comes down and inclined track and then around a vertical loop of radius R. At what height h above the top of the loop must its center of gravity be at the start in order that it may press against the bottom of the loop with K times the force that it presses against the top? Consider two cases,
(a) Rolling without slipping (neglect frictional loss);
(b) Sliding without rolling.

199. *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Projectiles are fired in a vertical plane at a given initial velocity but varying angles of elevation θ. Of all the pairs of trajectories for θ and $90° - \theta$, where $\theta < 45°$, which give the same range R_θ, show that there is only one pair such that the point P of maximum height attained for θ is the focus of the trajectory at $90° - \theta$. Find the value of θ for which this is true. (Consider trajectories in a vacuum under the influence of gravity).

200. *Proposed by Leon Bankoff, Los Angeles, California.*

Chords CD and EF are perpendicular to the diameter AB of a circle. Show that the radical axis of the circle with center at A of radius AC and the circle with center at B of radius BE is a line which is equidistant from CD and EF.

ERRATA. Problem **187** [Jan. 1953] should read $g(x) = 2 \text{ arc sin } x/\sqrt{x^2 + a^2}$

SOLUTIONS

A CONDITIONAL SERIES

97. [March 1951] *Proposed by Bruce Kellogg, Massachussets Institute of Technology.*

Let $\{i_n\} = i_1, i_2, i_3, \ldots$ be a sequence of real numbers such that $\lim_{n\to\infty} i_n = 1$ and $i_n > 1$ for all n. Does the infinite series

$$\sum_{n=1}^{\infty} {}^{1}/_{n}\, i_n$$

converge or diverge, or does the divergence or convergence depend upon the sequence $\{i_n\}$?

Solution by R. M. Foster and M. S. Klamkin, Polytechnic Institute of Brooklyn. Let

$$i_n = 1 + \frac{r \log \log n}{\log n}$$

where the logs are natural logarithms, then

$$\sum {}^{1}/_{n} i_n = \sum {}^{1}/_{n} (\log n)^2$$

Thus if r (constant) > 1 the series converges, and $r \leq 1$ the series diverges. See the solvers' note on the convergence of p-series in the *American Mathematical Monthly*, Nov. 1953, page 625.

A CRYPTARITHM

175. [September 1953] *Proposed by Jack Winter, Venice, California.*

Reconstruct the following cryptarithm given that each of the letters represents a distinct digit.

$$\begin{array}{rrr}
D & C & A \\
AA & BC & CB \\
** & & \\
* & ** & \\
* & ** & \\
 & ** & ** \\
 & ** & **
\end{array}$$

I. *Solution by R. M. Swesnik, Dallas, Texas.* D must be a digit such that $D \geq 4$. After $**$ is subtracted from AA the digit remaining must be either 6 or 8. It is readily seen that 6 is impossible. Now if $*$ is 8 then D must be 5 or 6. If 6 is used we are led to a contradiction. Inserting 5 for D it becomes apparent that A is 3 and the problem then is:

$$\begin{array}{rrr}
5 & C & 3 \\
33 & BC & CB \\
25 & & \\
8 & BC & \\
* & ** & \\
 & ** & CB \\
 & ** & *9
\end{array}$$

This implies that B must be 9. Then

$$\begin{array}{r} 89C \\ \underline{8{*}{*}} \\ {*}{*} \end{array} \quad \text{and} \quad \begin{array}{r} 10C \\ \underline{C} \\ 8{*}{*} \end{array}$$

from which it is obvious that $C = 8$ and the solution is:

$$\begin{array}{r} 5\ \ 8\ \ 3 \\ \sqrt{33\ 98\ 89} \\ \underline{25} \\ 8\ 98 \\ \underline{8\ 64} \\ 34\ 89 \\ \underline{34\ 89} \end{array}$$

II. *Solution by Leon Bankoff, Los Angeles, California.* In keeping with the spirit of the original cryptarithm and without impairing the uniqueness of the solution, the proposal could have been offered with greater economy as follows:

$$\begin{array}{r} D\ \ C\ \ A \\ \sqrt{AA\ {*}{*}\ {*}B} \\ \underline{{*}{*}\phantom{\ {*}{*}\ {*}B}} \\ {*}\ {*}{*}\phantom{\ {*}B} \\ \underline{{*}\ {*}{*}\phantom{\ {*}B}} \\ {*}{*}\ {*}{*} \\ \underline{{*}{*}\ {*}{*}} \end{array}$$

The key to the solution is that D^2 is the largest square less than AA and that B is the terminal digit of A^2.

Since each letter represents a distinct digit, we see that $A \neq B \neq 0, 1, 5, 6$. Also, $A \neq D \neq 9$.

For $A = 2$, we have the contradiction $B = D = 4$.
For $A = 4$, we have $B = D = 6$ contrary to hypothesis.
For $A = 7$, $D^2 = 64$ requiring a two digit first difference.
For $A = 8$, $D = 9$ and $B = 4$. Then the second product requires $C = 4$ with the untenable equality $B = C$.
For $A = 3$, we have $B = 9$ and $D = 5$, which leads to a valid solution when $C = 8$.

Also solved by *Norman Anning, Alhambra, California; Harvey H. Berry, Cincinnati, Ohio; Fred H. Bloedow, Waukesha, Wisconsin; Ben B. Bowen, Vallejo College, California; Bernice Brown, Santa Monica, California; W. O. Buschman, Portland, Oregon; Walter B. Carver, Cornell University; Monte Dernham, San Francisco, California; R. W. Gross and D. E. Freeland (Jointly) Purdue University; Richard*

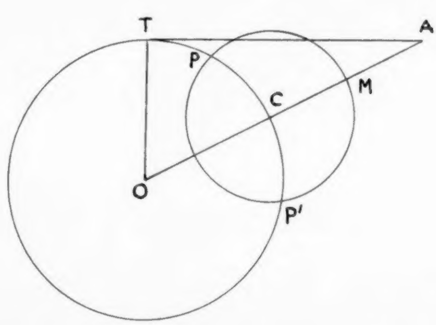
K. Guy, University of Malaya, Singapore; Herbert R. Leifer, Pittsburgh, Pennsylvania; William C. Lordan, Wesleyan University; J. H. Means, Huston-Tillotson College; Erich Michalup, Caracas, Venezuela; Francis L. Miksa, Aurora, Illinois; George R. Mott, Mineola, New York; Paul J. Orth, Missouri School of Mines; P. W. Allen Raine, Newport News, Virginia; Lawrence A. Ringenberg, Eastern Illinois State College; William Small, Rochester, New York; C. W. Trigg, Los Angeles City College (Five different solutions), Hazel Schoonmaker Wilson, Jacksonville State College and the proposer.

PENTAGON CONSTRUCTION

176. [September 1953] *Proposed by W. R. Ransom, Tufts College.*

Let TA, length $2(OT)$, be a tangent at T to the circle whose center is at O. Draw AO, cutting the circle at C. Let M be the midpoint of AC. With C as center draw a circle through M cutting the given circle at P and P'. Prove that this construction gives PP' the side of a regular pentagon inscribed in the circle whose radius is OT.

Solution by Hazel Schoonmaker Wilson, Jacksonville State College. Let $OT = r$. Then $AT = 2r$, and $AO = r\sqrt{5}$; $CA = r(\sqrt{5} - 1)$. Next $CP = CP' = CM = r(\sqrt{5} - 1)/2$. This is the side of a regular decagon inscribed in a circle of radius r. Since OC bisects the arc PP' it also bisects PP' and the central angle POP'. Hence PP' is the side of a regular pentagon inscribed in the circle whose radius is OT.



Also solved by *Norman Anning, Alhambra, California; Leon Bankoff, Los Angeles, California; Fred H. Blaedow, Waukesha, Wisconsin; Walter B. Carver, Cornell University; C. C. Carico, Los Angeles State College; O. L. Dunn, Vincennes, Indiana; Richard K. Guy, University of Malaya, Singapore; John Jones Jr., Mississippi Southern College; P. W. Allen Raine, Newport News, Virginia; William Small, Rochester, New York; C. W. Trigg, Los Angeles City College* and the proposer.

A MAP OF THE UNIT CIRCLE

177. [September 1953] *Proposed by Murray S. Klamkin, Polytechnic Institute of Brooklyn.*

If

$$w = z^n + a_1 z^{n-1} + \dots + a_n + b_1/z + b_2/z^2 + \dots + b_r/z^r$$

maps into $|w| = 1$ for $|z| = 1$, show that $a_n = b_r = 0$; $n = 1, 2, 3, \dots$ and $r = 1, 2, 3, \dots$.

I. *Solution by Alfredo Jones, University of Notre Dame.* If

$$w = z^n + a_1 z^{n-1} + \dots + a_n + b_1/z + \dots + b_r/z^r$$

maps $|z| = 1$ into $|w| = 1$ then

$$w' = w \cdot z^{r+1} = z^{n+r+1} + a_1 z^{n+r} + \dots + b_r z$$

also satisfies that condition. The area of the image of the unit circle by w' will be $A = k\pi$ where k is the number of times w' traverses $|w| = 1$ while z traverses $|z| = 1$. Now $k \le n + r + 1$ as k is also the number of zeros of w' for $|z| < 1$, which is less than or equal to $n + r + 1$. Writing $z = R\, e^{i\theta}$ this area is also:

$$A = \pi \int_0^{2\pi} zf'(z)\, \overline{f(z)}\, d\theta = \pi\, [n + r + 1 + (n+r)\ |a_1|^2 + \dots + |b_r|^2]$$

But this implies that $a_1 = a_2 \dots = b_r = 0$.

II. *Solution by Walter B. Carver, Cornell University.* We may write:

$$w = \frac{z^{n+r} + a_1 z^{n+r-1} + \dots + a_n z^r + b_1 z^{r-1} + \dots b_{r-1} z + b_r}{z^r}$$

If $|z| = 1$, $\bar{z} = 1/z$. Then

$$\bar{w} = \frac{\bar{b}_r z^{n+r} + \bar{b}_{r-1} z^{n+r-1} + \dots + \bar{b}_1 z^{n+1} + \bar{a}_n z^n + \dots + \bar{a}_1 z + 1}{z^n}$$

Also if $|w| = 1$, $w\bar{w} = 1$ and hence whenever $|z| = 1$ we have

$$z^{n+r} = (z^{n+r} + a_1 z^{n+r-1} + \dots + b_r)(\bar{b}_r z^{n+r} + \bar{b}_{n-1} z^{n+r-1} + \dots$$
$$\dots + \bar{a}_1 z + 1)$$

for all z on the unit circle. Hence this equation must be an identity and we can equate coefficients of like powers of z on the two sides of the equation. Thus (for $n \geq r$) we have:

$$\begin{array}{ll}
z^0: & b_r = 0 \\
z^1: & b_r a_1 + b_{r-1} = 0, \quad b_{r-1} = 0 \\
z^3: & b_r \bar{a}_2 + b_{r-1}\bar{a}_1 + b_{r-2} = 0, \quad b_{r-2} = 0 \\
\cdot & \\
\cdot & \\
\cdot & \\
z^{r-1}: & b_r \bar{a}_{r-1} + b_{r-1}\bar{a}_{r-2} + \dots + b_1 = 0, \quad b_1 = 0 \\
z^r: & b_r \bar{a}_r + b_{r-1}\bar{a}_{r-1} + \dots + a_n = 0, \quad a_n = 0 \\
\cdot & \\
\cdot & \\
\cdot & \\
z^n: & b_r \bar{a}_n + b_{r-1}\bar{a}_{n-1} + \dots + a_r = 0, \quad a_r = 0 \\
z^{n+1}: & b_r \bar{b}_1 + b_{r-1}\bar{a}_n + \dots + a_{r-1} = 0, \quad a_{r-1} = 0 \\
\cdot & \\
\cdot & \\
\cdot & \\
z^{n-r-1}: & b_r \bar{b}_{r-1} + b_{r-1}\bar{b}_{r-2} + \dots + a_1 = 0, \quad a_1 = 0 \\
z^{n+r}: & b_r \bar{b}_r + b_{r-1}\bar{b}_{r-1} + \dots + 1 = 1
\end{array}$$

For higher powers of z, from

$$z^{n+r+1} \text{ to } z^{2n+2r},$$

we would have merely the conjugates of the above equations, giving

$$\bar{a}_1 = 0, \dots \bar{a}_n = 0, \quad \bar{b}_1 = 0, \dots \bar{b}_r = 0.$$

Slight obvious changes in the set of equations would be necessary if $n < r$.

Also solved by the proposer.

A PYTHAGOREAN EQUATION

178. [September 1953] *Proposed by Pedro A. Piza, San Juan, Puerto Rico.*

Find prime numbers x, y and z, $2000 > z > y > x > 1$ satisfying the Pythagorean equation $(y + 4z - x)^2 = (y + 4z)^2$.

Solution by H. M. Feldman, St. Louis, Missouri. The above

equation is equivalent to $(3y)^2 = x(2y + 8z - x)$. Since it is required that y be prime it is obvious that 3 is the only value that x can assume. This leads to the new equation $3y^2 - 2y + 3 = 8z$. Again, since y is a prime the left hand member of this new equation must be divisible by 8 because the right hand member is. Now it is clear that we must have $y = 4 + n - 1$ and $z = 6n^2 - 4n + 1$.

Thus with $x = 3$ we get the following pairs for (y, z): $(7, 17)$; $(11, 43)$; $(19, 131)$; $(23, 193)$; $(31, 353)$; $(43, 683)$; $(67, 1667)$ and $(71, 1873)$.

It is interesting to note that the next prime pair in this set is $(131, 6403)$.

Also solved by *Leon Bankoff, Los Angeles, California; Bonnie Baker, University of North Carolina; Walter B. Carver, Cornell University; Nathaniel Grossman, West High School, Aurora, Illinois (Partially); Richard K. Guy, University of Malaya, Singapore; Francis L. Miksa, Aurora, Illinois; George R. Mott, Mineola, New York; Lawrence A. Ringenberg, Eastern Illinois State College; C. W. Trigg, Los Angeles City College* and the proposer.

THE INTERRUPTED STAG PARTY

179. [September 1953] *Proposed by John M. Howell, Los Angeles City College.*

Seven men are in a room when a fire breaks out and the lights go out. What is the probability that exactly four get their own hats?

Solution by Bernice Brown, Santa Monica, California. Let P_4 be the probability that a particular set of four men, say A, B, C and D get their own hats. Then

$$P_4 = \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{4} .$$

But this event can happen in ${}_7C_4$ ways. Therefore the probability that some set of four get their own hats is $35P_4$.

Now the probability that exactly four men get their own hats equals the product of the probabilities that some four get their own hats and that the remaining three do not get their own hats.

The probability that none of the remaining three get their own hats is, $Q_3 = 1/3$. Therefore the probability that exactly four men get their own hats is

$$P(4) = 35 \cdot \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{72} .$$

In general if n is the number of men (and hats) and r is the number of men getting their own hats,

$$P(r) = \frac{1}{r!} \sum_{i=0}^{N-R} \frac{(-1)^i}{i!}$$

which reduces in our case to

$$P(4) = \frac{1}{4!} \sum_{i=0}^{3} \frac{(-1)^i}{i!} = {}^1/_{72}$$

Also solved by *Walter B. Carver, Cornell University; Monte Dernham, San Francisco, California; W. W. Funkenbusch, Michigan College of Mining and Technology, Sault Ste. Marie Branch; Richard K. Guy, University of Malaya, Singapore; Lawrence A. Ringenberg, Eastern Illinois State College; William Small, Rochester, New York;* and the proposer.

Dernham made the conjecture that as the number of men increases without limit, that the successive probabilities that no one gets his own hat will converge to e^{-1}. The proposer pointed out that this is proved in *Probability Theory and Its Applications*. Vol. 1, by Feller on page 67.

A PELL-FERMAT PROBLEM

180. [September 1953] *Proposed by Leon Bankoff, Los Angeles, California.*

In a right triangle ABC whose sides are integers, $AC > BC > AB$. The bisector of the angle ABC meets AC at D. E is the projection of A upon BD and F is the midpoint of AC. If $EF = 49$, find AB, BC and AC.

Solution by Lawrence A. Ringenberg, Eastern Illinois State College. Extend AE to meet BC in G. Then $AE = EG$, $GC = 98$, and the problem reduces to finding positive integral solutions of $x^2 + (x + 98)^2 = y^2$. Suppose (x, y) is a solution with x odd, then $x = 2z + 1$, $8z^2 + 400z + 9802 = y^2$, $y = 2w$, w an integer, $4z^2 + 200z + 4901 = 2w^2$. This last equation is impossible since its members are numbers of different parity. Therefore x and y are even. Setting $x = 2a$, $y = 2b$, we have $a^2 + (a + 49)^2 = b^2$. Let (r, s, t) be any primitive Pythagorean triple in which $r^2 + s^2 = t^2$, $r + d = s$, where $d = 1, 7$, or 49. Let $e = 49/d$. Then (AB, BC, AC) $(2er, 2es, 2et)$ is one of the possible solutions.

Using Pythagorean generators we list several solutions:

u	v	e	x	$x + 98$	y
2	1	49	294	392	490
3	2	7	70	168	182
4	1	7	112	210	238
5	2	49	1960	2058	2842
6	5	1	22	120	122
8	3	7	672	770	1022
9	4	7	910	1008	1358
12	5	49	11662	11760	16562
13	4	1	208	306	370
19	8	7	4158	4256	5950

Also solved by *Norman Anning, Alhambra, California; Ben B. Bowen, Vallejo College, California; Walter B. Carver, Cornell University; O. L. Dunn, Vincennes, Indiana; Richard K. Guy, University of Malaya, Singapore; John Jones Jr., Mississippi Southern College; Sam Kravitz, East Cleveland, Ohio; Francis L. Miksa, Aurora, Illinois; Hazel Schoonmaker Wilson, Jacksonville State College,* and the proposer.

Anning provided the following references to the Pell-Fermat equation: Advanced Algebra by Barnard and Child, 1939; Chrystal's Algebra, 1906, Part II and footnote on p 152; Higher Algebra by W. L. Ferrar, 1948, p 287; Introduction To The Theory of Numbers by Hardy and Wright, 1938, Chapter X and Die Lehre von de Kettenbruchen by Perron, 1929.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 103. By mental arithmetic or judicious guessing find

$\sqrt{12345678987654321}$ [*Submitted by Leo Moser*]

Q 104. Factor

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1+x & 1+x & 1 & 1+x \\ 1 & 1+x & 1+x & 1 \\ 1 & 1 & 1+x & 1+x \end{vmatrix}$$

[*Submitted by M. S. Klamkin*]

Q 105. Sum the series $1 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 4,\ \cdots,\ n(n+1)$ [*Submitted by Fred Marer*]

Q 106. Find the center of gravity of a semicircular area. [*Submitted by M. S. Klamkin*]

Q 107. The three digits of a three digit number are in geometric progression. If the digits are reversed, the new number is 594 larger than the original number. What is the original number? [*Submitted by John M. Howell*]

Q 108. Find all the integral solutions of $x(x + 1)(x + 2)(x + 3) + 1 = y^2$. [*Submitted by M. S. Klamkin*]

Q 109. If A, B and C are three vectors originating from a common point, prove that $(A \times B) + (B \times C) + (C \times A)$ is a vector perpendicular to the plane determined by the terminal points of A, B and C. [*Submitted by M. S. Klamkin*]

ANSWERS

A 103. Solution *111,111,111*

A 104. When $x = 0$, the four rows are identical so x^3 is a factor. The sum of each of the rows is $4 + 2x$ which is therefore a factor. Thus $D = A\,x^3(x + 2)$, where A is a constant. Next x^4 appears in only two terms, each positive, so $A = 1 + 1$. Hence $D = 2x^3(x + 2)$

A 105.

$$1.2\ 2.3\ \ldots\ n(n+1) = 1(1+1)\ 2(2+1)\ 3(3+1)\ \ldots\ n(n+1)$$
$$= 1^2 + 2^2 + 3^2 + \ldots + n^2 + 1 + 2 + 3 + \ldots + n = \Sigma n^2 + \Sigma n$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(n+2)}{3}$$

A 106. Clearly the C.G. falls on the radius which is perpendicular to the diameter at a distance $\bar{y}$ from the diameter. When the area is rotated about the diameter a spherical volume is generated. Hence by Pappus' Theorem we have

$$2\pi\bar{y}(\pi r^2/_2) = 4\pi r^3/_3 \quad \text{so } \bar{y} = 4r/_{3\pi}$$

A 107. The only possible three digit numbers whose digits are in G. P. with ascending terms are *1 2 4 ; 139 ; 248*. Obviously the only one which satisfies the conditions is *248*.

A 108. Since the left hand side of the equation is a perfect square, all integral values of x will satisfy the equation.

A 109. Since $(A \times B) + (B \times C) + (C \times A) = (B - A) \times (C - B)$ the statement follows immediately.

FALSIES

A falsie is a problem for which a correct solution is obtained by illegal operations, or an incorrect result is secured by apparently legal processes. For each of the following falsies can you offer an explanation? Send in your favorite falsies.

F 14. A lazy student had to solve the simultaneous equations:

$$\frac{x - 2}{y - 1} = \frac{3}{5} \quad \text{and} \quad \frac{x - 1}{y - 2} = 1$$

He disregarded the second equation and solved the first by the following steps:

$$\frac{x - 2}{y - 1} = \frac{3}{5} \quad \text{gives} \quad \frac{x}{y} - \frac{2}{1} = \frac{3}{5}$$

then

$$\frac{x}{y} = \frac{2}{1} + \frac{3}{5} = \frac{2 + 3}{1 + 5} = \frac{5}{6} \quad \text{so} \quad x = 5,\ y = 6.$$

[*M. Kraitchik, Mathematical Recreations, Norton (1932), p 42.*)

F 15. Let z be a complex number such that $\tan z = i$. Then

$$\tan(z + w) = \frac{i + \tan w}{i - \tan w} = i$$

Thus the tangent of every complex number is i. [*Submitted by M. S. Klamkin*]

EXPLANATIONS

E 14. Obviously, the equation

$$\frac{x - a}{y - b} = \frac{c}{d}$$

is satisfied by $x = a + c$, $y = b + d$. Then

$$\frac{x - m}{y - n} = k$$

is also satisfied if k is chosen to equal

$$\frac{a + c - m}{b + d - n}.$$

E 15. The fallacy lies in the fact that no z exists such that $\tan z = i$.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the ***MATHEMATICS MAGAZINE****, lists of new books, and book reviews.*

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Andreas Speiser, Elemente Der Philosophie Und Der Mathematic. Verlag Birkhaeuser, Basel, 1952; 115 pp; 11.45 Swiss francs.

A prelude and a fugue, separated by a chapter devoted to remarks on the former and preparation for the latter, give the basic elements of Hegel's philosophy of science, with the critical comments and the modifications, which are dictated by mathematical considerations. The author's profound understanding of classical and of modern philosophy make it possible for him to indicate the essential defects in Hegel's philosophy. Discrete and continuous, being and not-being, growth and decay, the thesis, antithesis and synthesis of the Hegel dialectics come in for thorough discussion. This book will be of interest chiefly to mathematicians with a taste for and a good training in philosophy, well acquainted with Hegel's writings. Encouraging is the following passage (p.38): "One should not be frightened by the barking manner in which Hegel gives his explanations. When he wrote his logic, Hegel was rector of the Nuremberg gymnasium and he was accustomed to speak to the seniors in his school."

Arnold Dresden

Book Review, *British Association Mathematical Tables, Vol. X, Bessel Functions, Part II* (Functions of Positive Integral Order). New York, Cambridge University Press, (1952) $11.00, xl + 255 pages.

Part I of the Bessel Functions Tables appeared in 1937 (reprinted in 1950) and dealt with functions of orders zero and one. It is stated in the acknowledgements of Part II that some of the tabulated data was prepared prior to 1936, so that one may infer that World War II and other causes delayed publication of Part II.

This volume is issued in the British Association Mathematical Tables series. However, in 1948 a decision was made to transfer certain table preparation functions from the British Association for the Advancement of Science to the Royal Society, and a grant-in-aid from the British Government to the Royal Society has helped with the financial aspects of publication. The publication of this volume completes work well underway in 1948, and this is the last (from the point of publication date) of the British Association Mathematical Tables. It is presumed that future tables will bear the imprint of the Royal Society Mathematical Tables.

The functions tabulated are $J_n(x)$ and $Y_n(x)$ which are two linearly independent solutions of Bessel's differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0 \tag{1}$$

and $I_n(x)$ and $K_n(x)$ which are two linearly independent solutions of the related equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + n^2)\, y = 0 \tag{2}$$

In some cases, for example near the singularities at the origin of $Y_n(x)$ and $K_n(x)$, it has been found convenient to tabulate

$$y_n(x) = x^n Y_n(x), \quad k_n(x) = x^n K_n(x), \tag{3}$$

and also

$$i_n(x) = x^n I_n(x), \quad e^{-x} I_n(x), \quad e^{x} K_n(x) \tag{4}$$

rather than the solution itself.

All in all, there are eight tables. There are four tables (one for each of the four solutions or an alteration thereof) for range on integral n from 2 to 20 inclusive, with range on x $0(0.1$ or $0.01)$ 10 (0.1) 20 or 25 to 8 decimals or 8 figures with second differences tabulated.

The other four tables (again one for each of the four solutions or an alteration thereof) are for range on integral n from 0 to 20 inclusive, with range on x $0(0.1)$ 20 or 25 to 10 decimals or ten figures without differences.

The two tables dealing with $J_n(x)$ give less data than is available in the Harvard Bessel function tables [*1*]. Also the data in the Cambi tables [2] is given to more decimal places. But the volume under review does give data on the other three types of Bessel functions (which are not included in the two tables above).

As is customary with British tabular publications, the tables are printed from type. This volume represents a valuable and useful addition to the stock of published tables; every research worker needing tabular data on Bessel functions should have a copy in his reference library.

Robert E. Greenwood

[*1*] *Annals of the Harvard Computation Laboratory,* vols. III to XIV, Harvard University Press, Cambridge, Mass.

[*2*] Enzo Cambi, *Eleven and Fifteen Place Tables of Bessel Functions of the First Kind, to all Significant Orders,* New York, 1948.

Theory of Perfectly Plastic Solids by William Prager and Philip G. Hodge, Jr.

This text presents a stimulating and excellently written introduction to the theory of perfectly plastic solids. The general elastic-plastic problem depends upon the solution of a system of non-linear partial differential equations with data prescribed along unknown boundaries. Due to the mathematical complexities of this problem, the authors have stressed the physical ideas in the theory. However, for problems of unrestricted plastic flow, as well as for elastic-plastic problems with rotational symmetry, the pertinent mathematical theory for the problems under discussion is furnished. Thus, this text should be particularly suitable for engineering students at the senior-graduate level.

The authors' choice of topics should provide the reader with a good picture of the present day theory of perfectly plastic bodies. In addition to some of the older theory, the authors have included considerable recent theory on stress discontinuities and limit analysis. Most of this theory has been developed by the Brown University group under the direction of the senior author.

In the introduction and chapter 1, the basic physical and mathematical ideas are developed. The first topic provides the reader with some knowledge of the various experimental stress-strain results in plasticity. In chapter 1, the stress and strain tensors, and the stress and strain deviation tensors are introduced. Further

the stress equilibrium relations, the various yield conditions and the stress-strain relations of Mises and Prandtl-Reuss are discussed.

Chapters 2 and 3 are mainly concerned with the flexure of beams and the torsion problem. By use of appropriate assumptions, the theory of the flexure problem reduces to a one-dimensional problem. Hence, when one or more axes of symmetry exist, a complete discussion of the elastic-plastic problem is possible. The authors consider the beam with rectangular cross section in detail. For beams of other cross section (and possessing an axis of symmetry), the reader is referred to the well known text of V. V. Sokolovsky (Theory of Plasticity, Moscow, 1946). In the torsion problem, the cases of the complete elastic and unrestricted plastic cross section are discussed in some detail. For the elastic-plastic problem, it is shown that the derivatives of the stress functions are continuous across the elastic-plastic boundary and the one dimensional problem of the circular cross section is completely solved. The inverse method of Sokolovsky is used in treating problems of oval cross section. Finally, the warping of cross sections and the relation between the Saint Venant-Mises and the Prandtl-Reuss theories are discussed.

In chapters 4 through 7, the authors discuss various aspects of the problem of plane strain. First, the authors consider problems with axial symmetry (one dimensional problems). By use of the assumption of incompressibility, a solution is obtained for the elastic-plastic problem under the Prandtl-Reuss stress-strain relation as well as the unrestricted plastic flow problem under the Mises stress-strain law. In chapter 5, the theory of unrestricted plastic flow in the general plain strain problem for the Mises stress-strain relations is considered. Here, the stress tensor is statically determined by the equilibrium conditions and the yield condition (although, as authors note, some of the boundary conditions may be furnished in terms of the displacement vector). In order to avoid the transformation theory for the stress tensor which is necessary in determining the principal stress directions and the directions of the critical shearing stress (the first and second shear lines or the slip lines), the Mohr circle is used. Various properties of the Hencky-Prandtl sets of slip lines are considered and a graphical method for constructing these lines is discussed. One important property of these lines should be noted: "the envelope of the shear lines of one family is a limiting line across which the shear lines of the other family cannot be continued". This result leads the authors to a consideration of limiting lines (or "lines of rupture"). Along such lines, the shear rate and one derivative of one of the normal stresses become infinite. In connection with this problem, the jump conditions for stresses along lines of stress discontinuity are examined and applications made

(continued on back of table of contents)